Multiplicities in Commutative Algebra

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Submitted to the graduate degree program in the
Department of Mathematics and the
Graduate Faculty of the University of Kansas
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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Date defended: 6/30/2016

The Dissertation Committee for Jared Grant Serio certifies
that this is the approved version of the following dissertation :

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Date approved: 7/21/2016

Abstract

This dissertation explores the notion of multiplicity and its generalizations within the theory of commutative algebra. In a Noetherian local ring $(R, \mathfrak{m})$, Hilbert-Samuel multiplicity is an invariant of $\mathfrak{m}$-primary ideals which has become ubiquitous in the literature. The Hilbert function $F_I(n)$ of such an ideal, I, measures the length of R/I^n and is equal to a polynomial, $P_I(n)$ for all sufficiently large n. The Hilbert-Samuel multiplicity of I is the leading coefficient of $P_I(n)$, whose degree is $d = \dim(R)$. One common form in which this is presented is as the limit:

$$e(I) = \lim_{n\to\infty} \frac{d!P_I(n)}{n^d} = \lim_{n\to\infty} \frac{d!\lambda(R/I^n)}{n^d} \tag{1}$$

Over the years, mathematicians have found ways to apply the power of this invariant outside the context of $\mathfrak{m}$-primary ideals through the proper definition of a sequence of lengths. For example, Buchsbaum-Rim multiplicity applies to submodules of free modules with finite colength, and j and ε-multiplicities use the local cohomology functor to give multiplicities of ideals with maximal analytic spread. General j and ε multiplicities are given by Ulrich and Validashti which absorb all these multiplicities into a theory of relative multiplicity of algebras.

Chapter 2 is dedicated to calculating the limits which give rise to Buchsbaum-Rim multiplicities. We provide a formula for the multiplicity of a module in terms of its summands. An example demonstrates that multiplicity, dimension and rank alone are insufficient to give the multiplicity of a specified power of the module. We perform some explicit calculations for the special case of the module $I \oplus J$. Finally, we give a

method of approximating the multiplicity of an arbitrary module using a direct sum of ideals.

In the following chapter, we examine lengths given by a filtration, rather than powers of an ideal. We consider filtrations as generalizations of both valuations and ideal powers and identify a class of Rees valuations as Noetherian filtrations. We define a general j-multiplicity for Noetherian filtrations and prove some of its characteristics.

Chapter 4 is dedicated to ideals of submaximal analytic spread which remain nonzero when we apply the local cohomology functor. We determine that $\lambda(H^0_{\mathfrak{m}}(I^n/I^{n+1}))$, as a sequence in n, is also a polynomial for all large n and explore the relationship between the degree of this polynomial and the analytic spread of the ideal.

The final chapter is joint work with Tony Se. We analyze subrings of two-variable polynomial rings generated over a field, k, by monomials. We explicitly describe the asympototic behavior of a system of parameters and relate this to the Cohen-Macaulay property of the ring. We also give an algorithm describing a k-basis for the quotient of the ring by this system of parameters. These results specialize to projective monomial curves.

Acknowledgements

As any reader knows, no body of work this size can be considered the result of a lone person's labor. Thus, the duty is upon me to give credit here to some of the others by whom this result was made possible.

Naturally my mentor, Dr. Daniel Katz, stands at the head of this list. Not only has his knowledge been a critical support of my forays into research, but his patience with my mistakes has been formidible.

All of the professors on my committee have helped me to grow as a student. I will always be grateful for the wisdom and graciousness shown by each. In particular, conversations with Hailong Dao were essential to the development of Chapter 4. I also thank my colleague Tony Se for his friendship and for his work with me on Chapter 5.

Thanks are due to my wife, Kati, for her support these last six years. Without her in my life, I may have completed this document more quickly, but I could never have accomplished so much.

Finally, I thank God for the small miracle by which my unfounded claims were never very far from reality.

Contents

Chapter 1

A History of Multiplicities in Commutative Algebra

As a preliminary matter, we will review the current state of multiplicity theory and some of its applications. We begin with the development of Hilbert-Samuel multiplicity, a numerical invariant of an $\mathfrak{m}$-primary ideal in a local ring. This invariant is typically calculated using the Hilbert function of an ideal and takes its modern form in a 1951 paper of Samuel [35]. While studying generalized Koszul complexes, Buchsbaum and Rim [5] identified an analogous invariant for finite length modules in 1964. This Buchsbaum-Rim multiplicity has come to be interpreted as a more direct generalization of Hilbert-Samuel multiplicity by defining the invariant with respect to modules of finite colength in a free module. In 1993, Achilles and Manaresi presented a multiplicity for ideals of maximal analytic spread in a local ring and showed that this new multiplicity still described the intersection of algebraic varities. Finally, Simis, Ulrich and Vasconcelos [36] define a relative multiplicity of algebras in 2001. Through the use of Rees algebras associated to ideals and modules, each of the previously defined multiplicities may be studied through this unified framework.

1.1 Hilbert-Samuel Multiplicity

Throughout the text, R will be used to denote a commutative Noetherian ring with identity. Most rings will be assumed to be local, in which case $\mathfrak{m}$ will denote the maximal ideal and k will denote the residue field. We use the notation $(R,\mathfrak{m})$ or $(R,\mathfrak{m},k)$ as a reminder of this convention. $\lambda_R(-)$ will indicate the length of a module over the ring R. The indication of the ring will frequently be suppressed when context makes this clear.

One of the most important properties of length is its additivity. That is, if we have an exact sequence of R-modules, $0 \to M_1 \to M_2 \to \ldots \to M_n \to 0$, then $\sum_{i=1}^{n}(-1)^i\lambda(M_i) = 0$. In particular, we will often deal with short exact sequences and use that if $0 \to M' \to M \to M'' \to 0$ is exact and $\lambda(M) < \infty$, then $\lambda(M) = \lambda(M') + \lambda(M'')$.

Many of the lengths with which we work are related to polynomial functions. We frequently find this well-known computational lemma of use.

Lemma 1.1 ([4], 4.1.2)**.** *Let $F : \mathbb{Z} \to \mathbb{Z}$ be a numerical function and $d \geq 0$ an integer. The following conditions are equivalent:*

(i) $\Delta^d F(n) = c, c \neq 0$ *for all* $n \gg 0$.

(ii) There exists a polynomial of degree d such that $P(n) = F(n)$ for all $n \gg 0$.

Here Δ is the difference operator $\Delta F(n) = F(n) - F(n-1)$ and Δ^d is the difference operator iterated d times. We frequently use the following formulation of this result. If $F(n)$ equals a polynomial of degree d for all $n \gg 0$, then $\sum_{i=1}^{n} F(i) = G(n)$ is equal to a polynomial of degree $d+1$ for $n \gg 0$ since $\Delta G = F$.

1.1.1 Graded and Local Rings

A ring is **graded** by a semigroup G if there exist $R_g \subset R$ for each $g \in G$ with the following properties:

(i) $R = \oplus_{g \in G} R_g$

(ii) if $a \in R_g$ and $b \in R_h$, then $ab \in R_{g+h}$

When these hold, R is referred to as G-graded. The most common index semigroups are $\mathbb{Z}$ and $\mathbb{N}$. It is worth noting that when R is a graded ring, R_0 is a subring since it is closed under addition and multiplication.

Example 1.2. $R = k[x,y]$ may be considered an $\mathbb{N}$-graded ring with R_i equal to the set of polynomials whose terms all have total degree i. This ring may also be considered an $\mathbb{N}^2$-graded ring in which the graded pieces are in one-to-one correspondence with monomials of R. Under the first grading, $x^3y^2 + x^2y^3$ would be in R_5 and under the second x^3y^2 and x^2y^3 would fall into different graded pieces, $R_{(3,2)}$ and $R_{(2,3)}$ respectively.

An element of a graded ring, R, is said to be **homogeneous** if it is contained in a single graded piece of R. An ideal of R is said to be **homogeneous** if it may be generated by homogeneous elements of R. If R is graded by $\mathbb{N}$ or $\mathbb{Z}$ and is generated over R_0 by elements of degree 1, we call R a **standard graded ring**. The polynomial ring above taken with the $\mathbb{N}$-grading described is standard graded. You may note in Example 1.2 that $x^3y^2 + x^2y^3$ is homogeneous in the first grading but not in the second.

A graded ring R is called a **graded local ring** if there is a unique homogeneous maximal ideal. This differs from the standard definition of 'local' since R may well have another maximal ideal which is not homogeneous. The standard example is again a polynomial ring over a field. If $R = k[x]$, then (x) and $(x-1)$ are both maximal ideals, though (x) is the only homogeneous maximal ideal. The behavior of these rings is similar enough to local rings that this nomenclature is ubiquitous.

Given a (non-graded) ring R and an ideal $I \subset R$, there are three graded rings which are frequently constructed in the study of multiplicity. Letting t denote a new variable, $R[It]$ is called the **Rees algebra** of I. The ring has a natural standard $\mathbb{N}$-grading by powers of t, giving $R[It]_n \cong I^n$ as R-modules. Quotient the Rees algebra by the ideal $IR[It]$ and we obtain what is known as the **associated graded ring** of I: $G(I) = R[It]/IR[It] \cong \oplus_{i=0}^{\infty} I^i/I^{i+1}$. A third critical construction is the **extended Rees algebra**, which proved pivotal in much of Rees' work: $\mathscr{R}(I) := R[It, t^{-1}]$. This is $\mathbb{Z}$-graded by powers of t with $R[It, t^{-1}]_n \cong R$ for any $n \leq 0$. In the extended Rees ring,

$(I\mathscr{R}(I))_n = (t^{-1}\mathscr{R}(I))_n$ for all $n \geq 0$, so that we may recover $G(I)$ as the quotient of $\mathscr{R}(I)$ by a principal ideal: $G(I) = R[It, t^{-1}]/t^{-1}R[It, t^{-1}]$.

Proposition 1.3 ([17], 5.1.4)**.** *Let R be a ring with* $\dim(R) = d$*. Let* $\mathscr{R}(I)$ *denote the extended Rees ring of an ideal I in R. Then* $\dim(\mathscr{R}(I)) = d+1$ *and* $\dim(\mathscr{R}(I)/t^{-1}\mathscr{R}(I)) = d$*. If I is contained in a prime P with* $\dim(R/P) = \dim(R)$*, then* $\dim(R[It]) = d$*. Otherwise,* $\dim(R[It]) = d+1$.

Example 1.4. Let $I = (3) \subset \mathbb{Z}$. $R[It] = R[3t]$, $\mathscr{R}(I) = R[t^{-1}, 3t]$. The graded pieces of $G(I)$ are each $\mathbb{Z}/3\mathbb{Z}$-modules: $G(I)_n = (3^n)/(3^{n+1})$. Each graded piece has length 1 as a $\mathbb{Z}/3\mathbb{Z}$-module.

Using these constructions, we interpret questions about multiplicity of ideals in a local ring as questions about standard graded local rings. The formal proof of results begins with a multiplicity of modules over a standard graded ring.

A module over a graded ring may also be graded. M is a **graded module** of an H-graded ring, R, if M may be written as $\oplus_{g\in H} M_g$ such that for every $a \in R_g, m \in M_h$ we find $am \in M_{g+h}$. In the following theorem on graded modules, we use the convention that the degree of the zero polynomial is -1 and the degree of a nonzero constant is 0.

Theorem 1.5 ([4], 4.1.6)**.** *Let M be a finitely generated graded module over a standard graded local Noetherian ring R with* R_0 *an Artinian subring. If* $\lambda(M_i)$ *is finite, then* $\lambda(M_n)$ *for large n is given by a polynomial in n of degree* $\dim(M) - 1$.

Proof. Let us proceed by induction on $\dim(M) = d$. We use a prime filtration of M; there exist a sequence of modules $\{A_0, \ldots, A_t\}$ such that $0 = A_0 \subset A_1 \subset \ldots \subset A_t = M$ and for each $0 \leq i < t$, there is a graded prime ideal p_i and a shift in the grading by a_i for which $A_{i+1}/A_i \simeq (R/p_i)(a_i)$. By additivity of λ, $\lambda(M_n) = \sum_{i=0}^{t-1} \lambda((R/p_i)(a_i)_n)$. If $\lambda(R/p_i)$ is eventually polynomial, then this polynomial has a positive leading coefficient and $\sum_{i=0}^{t-1} \lambda(R/p_i)$ will be a polynomial whose degree is the maximum of the dimensions of R/p_i. Hence we need only prove the statement for $M = R/p$.

If $\dim(R/p) = 0$, then p is the unique homogeneous maximal ideal and $(R/p)_n = 0$ for $n > 0$.

If $\dim(R/p) > 0$, then there exists $x \in R/p$ homogeneous of positive degree. Since R is standard graded, we may choose an x of degree 1. Now we consider the exact sequence given by multiplication with x:

$$0 \longrightarrow (R/p)(-1) \xrightarrow{x} R/p \longrightarrow R/(x)+p \longrightarrow 0$$

This gives $\lambda((R/p)_n) - \lambda((R/p)_{n-1}) = \lambda((R/(x)+p)_n)$. Note that $\dim(R/(x)+p) = d-1$. By induction, we assume that $\lambda((R/(x)+p)_n)$ is a polynomial of degree $d-2$ for $n \gg 0$. Name this polynomial $L(n)$. Now $\lambda((R/p)_n) = \sum_{i=k}^{n} L(i) + c$ for some constants k, c and Lemma 1.1 completes the proof. $\square$

Given a graded module M whose graded pieces have finite length, we define the **Hilbert function** of M as $F(i) := \lambda(M_i)$. By Theorem 1.5, $i \gg 0$ gives $F(i) = P(i)$ for some polynomial $P(i)$ of degree $d-1$. We also consider the cumulative Hilbert function formed by $\sum_{i=0}^{n} \lambda(M_i)$. For large n, we may consider $\sum_{i=1}^{n} F(i)$ to be a fixed constant added to a sum of polynomials. By Lemma 1.1, this cumulative function is eventually polynomial of degree d. Either of these may be called the Hilbert polynomial of the module M. When the leading coefficients of these polynomials are normalized with multiplication by $(d-1)!$ and $d!$ respectively, the coefficients are equal. This coefficient is a critical invariant called the graded multiplicity of M, denoted $e_0(M)$ or simply $e(M)$. More generally, we may define the Hilbert coefficients of M by $e_0(M), \ldots, e_d(M)$ where

$$\sum_{i=0}^{n} \lambda(M_i) = e_0(M)\binom{n+d}{d} - e_1(M)\binom{n+d-1}{d-1} + \ldots + (-1)^{d-1} e_{d-1}(n+1) + (-1)^d e_d(M)$$

When we are only concerned with e_0, we often present it as a limit:

$$e(M) = \lim_{n \to \infty} \frac{d! \sum_{i=1}^{n} \lambda(M_i)}{n^d}$$

Example 1.6. Let k be a field and let $R = k[x,y,z]$ be given the standard $\mathbb{N}$-grading. Each graded piece of R is an R_0-module, so R_n is a vector space over k whose length is the number of monomials of total degree n. For three variables, this is $\binom{n+2}{2} = \frac{n^2+3n+2}{2}$. In this case, the Hilbert function is equal to the Hilbert polynomial for all n. The multiplicity of R as a graded R-module is 2! times the leading coefficient: $e(R) = 1$.

Every generalization of multiplicity may be formulated in terms of a graded multiplicity, with various difficulties arising in the analysis of the appropriate graded rings and modules. The simplest modules of finite length are found by examining $\mathfrak{m}$-primary ideals in a local ring. The following proposition is well-known.

Proposition 1.7. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and I an ideal. Then R/I has finite length over R if and only if I is $\mathfrak{m}$-primary.*

Proof. Without loss of generality, we may replace R/I by R and I by (0). The statement becomes R has finite length if and only if (0) is $\mathfrak{m}$-primary. A module has a finite composition series exactly when that module satisfies ascending and descending chain conditions. Hence R has finite length if and only if R is Artinian if and only if (0) is $\mathfrak{m}$-primary. □

Let I be an $\mathfrak{m}$-primary ideal in a local ring R. We consider the standard graded local ring $\mathscr{R}(I)$. Its homogeneous maximal ideal is given by $\mathscr{M}$ with $\mathscr{M} \cap R = \mathfrak{m}$ and $\mathscr{R}_i \subset \mathscr{M}$ for $i > 0$. The associated graded ring of I is a graded module over this ring. By Theorem 1.5, $\lambda(I^n/I^{n+1})$ is eventually a polynomial of degree $\dim(R) - 1$. It follows, as before, that $\lambda(R/I^n) = \sum_{i=1}^{n} \lambda(I^{i-1}/I^i)$ is eventually a polynomial of degree $\dim(R)$. This construction produces what is known as the Hilbert-Samuel polynomial of the ideal I with leading coefficient $e(I)$.

In developing the theory of multiplicity, it is helpful to define it more generally, in terms of R-modules.

Definition 1.8 ([17], 11.1.5)**.** Let R be a local ring of Krull dimension d, I an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. The **Hilbert-Samuel multiplicity** of I with respect to M is

$$e(I, M) := \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \lambda(I^n M / I^{n+1} M) = \lim_{n\to\infty} \frac{d!}{n^d} \lambda(M/I^n M)$$

We recover the standard multiplicity $e(I)$ by assigning $M = R$. We conclude our introduction of multiplicity by observing its behavior on exact sequences.

Proposition 1.9 ([17], Theorem 11.2.3). *Let $(R,\mathfrak{m})$ be local and let $I \subset R$ be an $\mathfrak{m}$-primary ideal. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely-generated R-modules. Then $e(I,M) = e(I,M') + e(I,M'')$.*

Proof. Let $P(n), P'(n), P''(n)$ be polynomials which asymptotically define the lengths of M/I^nM, M'/I^nM' and M''/I^nM'', respectively. By identifying M' with its image in M, we may consider $M'' = M/M'$ and rewrite the quotient M''/I^nM'' as $M/(M'+I^nM)$.

$$\begin{aligned} P(n) =& \lambda(M/I^nM) \\ =& \lambda(M/(M'+I^nM)) + \lambda((M'+I^nM)/I^nM) \\ =& P''(n) + \lambda(M'/(I^nM \cap M')) \end{aligned}$$

By the Artin-Rees lemma, there exists a constant c such that for $n > c$, $I^nM' \subset M' \cap I^nM \subset I^{n-c}M'$. This gives $P'(n) \geq \lambda(M'/(M' \cap I^nM)) \geq P'(n-c)$ for all sufficiently large n. Hence $P(n) - P''(n)$ is a polynomial whose leading term is identical to that of $P'(n)$.

The leading coefficients of $P(n), P'(n), P''(n)$ are $\frac{e(M)}{d!}, \frac{e(M')}{d!}, \frac{e(M'')}{d!}$, respectively. Therefore, $e(M) = e(M') + e(M'')$. □

1.1.2 Integral Closure and Reductions

One of the most famous results of David Rees establishes a correspondence between the Hilbert-Samuel multiplicity of an ideal and its integral closure. We shall give a quick introduction to integral closure and reductions. A more complete treatment of the subject is available in [17].

Definition 1.10 ([17], 1.1.1). Let R be a ring. An element $r \in R$ is said to be **integral** over an ideal I if there exist elements $a_i \in I^i$ such that

$$r^n + r^{n-1}a_1 + r^{n-2}a_2 + \ldots + ra_{n-1} + a_n = 0 \tag{1.1}$$

We refer to this as an equation of integral dependence. We define the integral closure of an ideal $\overline{I} := \{r \in R | r \text{ is integral over } I\}$. This set is again an ideal of R and is integrally closed in the sense that $\overline{\overline{I}} = \overline{I}$.

Closely related to the notion of integral closure is the reduction of an ideal.

Definition 1.11 ([17], 1.2.1)**.** Let $J \subset I$ be ideals. We say that J is a **reduction** of I if there exists an integer t such that $JI^t = I^{t+1}$.

Proposition 1.12 ([17], 1.2.2)**.** *Let J be an ideal in a ring R. Then an element $r \in R$ is integral over J if and only if J is a reduction of $J + (r)$.*

Proof. Suppose r is integral over J. We have $r^k = r^{k-1}a_1 + r^{k-2}a_2 + \ldots + a_k$ for some $a_i \in J^i$. We will show that $(J+(r))^k = J(J+(r))^{k-1}$.

$$\begin{aligned}(J+(r))^k &= \sum_{i=0}^{k} J^{k-i}(r)^i \\ &= J\left(\sum_{i=0}^{k-1} J^{k-i-1}(r)^i\right) + (r)^k = J(J+(r))^{k-1} + (r^k)\end{aligned}$$

The equation of integral dependence demonstrates $r^k \in J(J+(r))^{k-1}$ so $(J+(r))^k = J(J+(r))^{k-1}$ as desired.

Suppose J is a reduction of $J+(r)$. There exists k such that $J(J+(r))^{k-1} = (J+(r))^k$. In particular, $r^k \in J(J+(r))^{k-1}$ so there exists an equation of integral dependence of r over J whose degree is k. □

Using this proposition and basic properties of reductions, it is easy to extend the result to finitely generated ideals. That is, for $J \subset I$ ideals and I finitely generated, J is a reduction of I if and only if $I \subset \overline{J}$.

The following is an important link between reductions and Rees rings.

Proposition 1.13 ([17], 8.2.1)**.** *Let R be a ring and let $J \subset I \subset R$ be ideals. Let I be finitely generated. J is a reduction of I if and only if $R[It]$ is module-finite over $R[Jt]$.*

Proof. Suppose $JI^{n-1} = I^n$ so that J is a reduction of I. Our rings are Noetherian, so I is finitely generated and a finite number of elements is needed to generate $It, I^2t^2, \ldots, I^{n-1}t^{n-1}$ over $R[Jt]$. Then for each $\alpha \in I^m$ for $m \geq n$, $\alpha \in J^{m-n+1}I^{n-1}$ which lies in the $R[Jt]$ module generated in degree i by I^it^i for $0 < i < n$.

Conversely, let $R[It]$ be module-finite over $R[Jt]$. In particular, let the $R[Jt]$ module generated by $x_1t^{p_1}, \ldots, x_kt^{p_k}$ be $R[It]$. Then for $n > \max_i(p_i)$,

$$\begin{aligned} I^n =& J^n + J^{n-p_1}x_1 + \ldots + J^{n-p_k}x_k \\ =& J(J^{n-1} + J^{n-p_1-1}x_1 + \ldots + J^{p_k-1}x_k) = JI^{n-1} \end{aligned}$$

□

We may also speak of the integral closure of rings. Let $R \subset S$ be rings. We say that an element $s \in S$ is **integral over** R if there exists an equation as in 1.1 with $a_i \in R$. When no overring is specified, the integral closure of the domain R is taken to be the integral closure of R in its field of fractions; that is, the localization of R at the prime ideal (0).

Rings and ideals generated by monomials provide the clearest examples of integral closure.

Example 1.14. Let $R = k[x,y]$, $I = (x^2, y^2)$. The equation $(xy)^2 + (xy) \cdot 0 - x^2 \cdot y^2 = 0$ demonstrates that xy is integral over I. The integral closure of I is (x^2, xy, y^2).

Example 1.15. Let $S = k[x^2, y^2] \subset R = k[x,y]$. The integral closure of S in R is R, since R is generated as a ring by elements integral over S. However, S is integrally closed as a ring since elements such as x, y, xy are not in the field of fractions of S.

Integral closures arise naturally in a variety of settings. The following theorem gives strong motivation for studying the multiplicity by connecting the two theories. A proper statement of Rees' Theorem requires a somewhat technical definition which will recur later in the work.

Definition 1.16 ([19], 1.1)**.** Let $(R, \mathfrak{m})$ be a local ring and let $\hat{R}$ be its $\mathfrak{m}$-adic completion. A ring R is **unmixed** if $\dim(\hat{R}) = \dim(\hat{R}/P)$ for any associated prime $P \subset \hat{R}$. R is **quasi-unmixed** if

$\dim(\hat{R}/P) = \dim(\hat{R})$ for any minimal prime $P \subset \hat{R}$.

Theorem 1.17 (Rees, [30])**.** *Let R be a quasi-unmixed local ring and let $J \subset I$ be $\mathfrak{m}$-primary ideals. $I \subset \overline{J}$ if and only if $e(I) = e(J)$.*

Example 1.18. Let $R = k[x,y]$ and $J = (x^3, y^3) \subset R$. Then $\overline{J} = (x^3, x^2y, xy^2, y^3)$. Any ideal between the two, like $I = (x^3, x^2y, y^3)$, has the same multiplicity as these ideals.

We may count $\lambda(R/(x^3, x^2y, xy^2, y^3)^n)$ as the number of monomials in two variables of degree less than 3n: $\sum_{i=0}^{3n-1} i + 1 = \sum_{i=1}^{3n} i = \frac{3n(3n+1)}{2} = \frac{9}{2}n^2 + \frac{3}{2}n$.

As a k-vector space, $R/(x^3, y^3)^n$ includes all monomials of degree less than $3n$. It also includes monomials $x^{3(n-i)-1}y^{3i+1}, x^{3(n-i)-2}y^{3i+2}, x^{3(n-i)-1}y^{3i+2}$ for $0 \leq i < n$. We may calculate $\lambda(R/(x^3, y^3)^n) = \lambda(R/(x,y)^{3n}) + 3n = \frac{9}{2}n^2 + \frac{9}{2}n$.

Any ideal I with $J \subset I \subset \overline{J}$ will have $\lambda(R/\overline{J}^n) \leq \lambda(R/I^n) \leq \lambda(R/J^n)$. Hence, the multiplicity of any such ideal is 9.

In the case of monomial ideals in a polynomial ring, the integral closure and multiplicity each have geometric interpretations which help to illuminate this connection. Let $R = k[x_1, \ldots, x_d]$. Each monomial in R may be considered a point in the lattice $\mathbb{N}^d$ by letting the i^{th} coordinate correspond to the power of x_i in the monomial. Let I be an ideal of R generated by monomials. We may consider the collection of points in $\mathbb{N}^d$ corresponding to all the monomials in the ideal. Define the Newton polyhedron of I to be the convex hull of these points.

Example 1.19. Let $I = (x^3, y^4) \subset k[x,y]$. In Figure 1.1, monomials in I correspond to the lattice points along or above the red line. The bounded face of the Newton polyhedron of I is in blue, so that any lattice points between the red and blue lines lie in $\overline{I} \setminus I$.

Now, the integral closure of I is described visually by all the lattice points of $\mathbb{N}^d$ which lie in the Newton polyhedron of I. Moreover, it is shown in [39] that the multiplicity of an $\mathfrak{m}$-primary ideal is given by the volume of the complement of its Newton polyhedron. In figure 1.1, this is the area of the blue and black triangle. This gives a simple proof in the monomial case that $J \subset I$ have equal

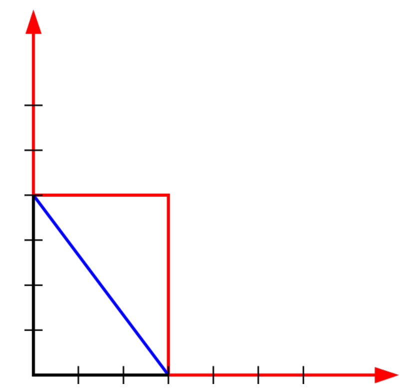


Figure 1.1: $I = (x^3, y^4)$

multiplicity exactly when $I \subset \overline{J}$. If their integral closures are different, then the complementary volumes must be different since the polyhedron for I strictly contains the polyhedron for J.

1.1.3 Valuations

We find the theory of Rees valuations helpful for studying multiplicity and integral closure. We now define a valuation and describe how to form the Rees valuation of an ideal.

Let K be a field whose multiplicative group $K \setminus \{0\}$ is K' and let G be a totally ordered abelian group. A **valuation** is a group homomorphism $v : K' \to G$ with the following properties:

(i) $v(x) + v(y) \geq \min(v(x), v(y))$

(ii) $v(xy) = v(x) + v(y)$

We are most interested in valuations which develop from rings. When R is a domain, any map $v : R \setminus \{0\} \to G$ with the given properties extends uniquely to a valuation on the fraction field of R. Thus, such a map is referred to as a valuation on R.

Example 1.20. Let $R = k[x,y,z]$. A rank one discrete valuation of R is a map $v : k(x,y,z) \setminus \{0\} \to \mathbb{Z}$. Let v be such a map with the additional assumption that the valuation of any polynomial equals the minimum valuation of its terms. In this case, we say that v is a monomial valuation and is determined by $v(x), v(y), v(z)$. If v is a monomial valuation with $v(x) = 2, v(y) = 3, v(z) = 4$, then $v(x^2y - 3y^3 + 8z^2) = \min(7,9,8) = 7$.

We may also consider (multiplicative) group homomorphisms $v : R \to G \cup \{\infty\}$ which satisfy

$v(x+y) \geq \min(v(x), v(y))$ and $v(xy) = v(x) + v(y)$. Since $v(0) = v(0 \cdot x) = v(0) + v(x)$ for any $a \in R$, it follows that $v(0) = \infty$ or $v(x) = 0$ for all $x \in R$. By extension, $v(x) = \infty$ for all x in some minimal prime $P \subset R$, so such homomorphisms correspond to valuations on R/P.

Definition 1.21 ([17], 6.1.8)**.** We say that two valuations, $v : K' \to G_v$ and $w : K' \to G_w$ are **equivalent** if there exists an order-preserving isomorphism $\phi : image(v) \to image(w)$ such that for all $\alpha \in K'$, $\phi(v(\alpha)) = w(\alpha)$.

In terms of discrete valuations, this equivalence identifies maps like $v : K' \to \mathbb{Z}$ and $2v : K' \to \mathbb{Z}$.

Let V be a subring of a field K such that for each $x \in K, x \in V$ or $x^{-1} \in V$. Then V is called a **valuation ring** of K. As a subring of a field, valuation rings are always domains. For each valuation v of K, we may produce a valuation ring by $V := \{a \in K | v(k) \geq 0\}$. Conversely, we have the following proposition.

Proposition 1.22 ([17], Proposition 6.2.3)**.** *Let V be a valuation domain with field of fractions K. Let K', V' be the respective multiplicative groups of units. Then there is a group homomorphism $v : K' \to K'/V'$ and a total order on K'/V' such that v is a valuation with value group K'/V'.*

If v is a valuation with valuation ring V, then the valuation produced by applying 1.22 is equivalent to v. Similarly, the natural valuation associated to a valuation ring V by 1.22 has V as its valuation ring. Thus sets of valuations and their corresponding valuation rings are often used interchangeably.

Of particular interest in our context are discrete valuation rings of rank one (DVRs).

Proposition 1.23 ([17], Proposition 6.4.4)**.** *Let R be a local domain, K its field of fractions and $R \neq K$. The following are equivalent:*

(i) R is a Noetherian valuation domain

(ii) R is a principal ideal domain

(iii) R is Noetherian and the maximal ideal is principal

(iv) R is Noetherian and there is no ring between R and K

(v) R is Noetherian, integrally closed and $\dim(R) = 1$

(vi) $\cap_n \mathfrak{m}^n = 0$ *and* $\mathfrak{m}$ *is principal*

(vii) R *is a valuation ring with value group isomorphic to* $\mathbb{Z}$*.*

Example 1.24. Let $R = k[x,y]_{(x,y)}$ and let $I = (x^2, y^3)$. Adjoin $\frac{x^2}{y^3}$ to form $S' := R[\frac{x^2}{y^3}] = k[x,y,\frac{x^2}{y^3}]$. Take S to be the integral closure of S' (in $k(x,y)$). We will claim $S = S'[\frac{x}{y}]$ by showing that $\frac{x}{y}$ is integral over S' and then that $S'[\frac{x}{y}]$ is integrally closed. First, $\left(\frac{x}{y}\right)^2 = y \cdot \frac{x^2}{y^3} \in S'$, so $\frac{x}{y} \in S$. Then $S'[\frac{x}{y}] = k[x,y,\frac{x}{y},\frac{x^2}{y^3}]$, but since $x = y \cdot (\frac{x}{y})$, x is not necessary as a generator of $S'[\frac{x}{y}]$ over k.

$$S'[\frac{x}{y}] = k[y,\frac{x}{y},\frac{x^2}{y^3}] \cong k[y,a,b]/(yb - a^2)$$

This ring is a domain with height 1 primes (y,a) and (b,a). Localization at (0) gives a field and localization at the height one primes gives a ring whose maximal ideal is principally generated by (a). Hence we have Serre's condition R_1. Being a complete intersection, $S'[\frac{x}{y}]$ satisfies S_2 and, thus, is normal. Therefore, $S = S'[\frac{x}{y}]$.

Now $IS = (y^3)S$ is principal and is contained in one of the height one primes: $P = (y,a)$. We have observed that S_P has a principal maximal ideal (a), so by Proposition 1.23 (iii) it is a DVR. Note that $\frac{x^2}{y^3}$ is a unit in S_P, $y = \frac{y^3}{x^2}(\frac{x}{y})^2$ and $x = \frac{y^3}{x^2}(\frac{x}{y})^3$. This DVR gives the monomial valuation on R with $v(x) = 3, v(y) = 2$.

Given any ideal I in a Noetherian ring, there exists a finite collection of DVRs which determine the integral closures of I and its powers. That is, $\exists V_1, \ldots, V_r$ discrete valuation rings of rank 1 such that for each i there is a minimal prime $P_i \subset R$ with $R/P_i \subset V_i \subset \kappa(P_i)$, where $\kappa(P_i)$ is the quotient field of R/P_i. Given these valuation rings we have

$$\overline{I^n} = \cap_{i=1}^r (I^n V_i \cap R) \tag{1.2}$$

These 'ideal valuations' were discovered and investigated by David Rees [29] and have since become known as **Rees valuations** of an ideal. If $\mathfrak{m}_i$ is the maximal ideal of V_i, then $Q_i = \mathfrak{m}_i \cap R$ is a prime ideal of R which is called the center of the Rees valuation.

The DVR in Example 1.24 is a Rees valuation of the ideal (x^2, y^3). In fact, Rees valuations in general may be constructed by following the process in that example.

Construction 1.25. Let R be a ring and $I \subset R$ an ideal.

- Choose a minimal prime P of R and let $\{g_1, \ldots, g_t\}$ be a minimal generating set of $I + P/P$.
- Adjoin fractions of the form I/g_k to the quotient ring as $S := (R/P)[I/g_k]$.
- Take the integral closure, $\overline{S}$, of S.
- Let V be the localization of $\overline{S}$ at a prime minimal over $g_k\overline{S}$.

V is the DVR of a Rees valuation of I.

Different Rees valuations may arise from the same ideal by selecting a different minimal prime of R for the quotient, a different generator of $(I+P)/P$ to invert or a different prime of $\overline{S}$ at which to localize.

Rees expressed multiplicity in terms of the Rees valuations. In describing this connection, we first wish to reduce the calculation of multiplicity to the calculation of multiplicity over a domain. The following theorem gives what is commonly called the 'associativity formula' for multiplicity.

Theorem 1.26 ([17], 11.2.4)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring, I an $\mathfrak{m}$-primary ideal, M a finitely generated R-module and Λ the set of primes P such that* $\dim(R/P) = \dim(M)$. *Then*

$$e(I,M) = \sum_{P \in \Lambda} e(I, R/P)\lambda(M_P)$$

Proof. We may replace R by $R/Ann(M)$ and thus consider $\dim(M) = \dim(R)$ and Λ the set of primes P with $\dim(R/P) = \dim(R)$.

Fix a prime filtration of M: $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ with $M_{i+1}/M_i \cong R/P_i$ for some prime P_i. Let $F_M(n), F_i(n)$ be the Hilbert polynomials of M, R/P_i, respectively. By 1.9, the leading coefficients are additive on short exact sequences. We have $0 \to M_i \to M_{i+1} \to R/P_i \to 0$, so $F_M(n) = \sum_i F_i(n)$. $F_i(n)$ does not contribute to the leading term if $\dim(R/P_i) < \dim(R)$, so the only terms which need be considered are the $P_i \in \Lambda$. Each P_i may not be distinct, so we must count the number of times each $P \in \Lambda$ appears in the filtration. Consider M_P, localized at $P \in \Lambda$. Since

P does not properly contain any prime ideals, P is the only prime of R_P. The number of times $e(I, R/P)$ is added to the leading term is $\lambda(M_P)$. □

The following lemma is a reformulation of [31], Lemma 9.36. In context it is part of an argument which extends its result to Noetherian filtrations of $\mathfrak{m}$-primary ideals. The more general result is cited in Theorem 3.26, after we have developed the background on Noetherian filtrations.

We use one new piece of notation: for a valuation v and an ideal I, we denote $\min(v(x)|x \in I)$ by $v(I)$.

Lemma 1.27 ([31])**.** *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension d. Let $I = (x_1, \ldots, x_d)$ be an $\mathfrak{m}$-primary ideal of R. Let $v_1, \ldots, v_s$ be the Rees valuations of I, and K_j the fraction field of v_j. Then, letting y_{ji} be the image of $\frac{x_i}{x_d}$ in K_j for $1 \leq i \leq d-1$, $\{y_{j1}, \ldots, y_{j(d-1)}\}$ is a transcendence basis for K_j over k. Furthermore, $e(I) = \sum_{j=1}^{s}[K_j : k(y_{j1}, \ldots, y_{j(d-1)})]v_j(I)$*

1.2 Buchsbaum-Rim Multiplicity

In their study of Koszul complexes, Buchsbaum and Rim [5] produced a multiplicity which could be applied to certain finite length modules over a local ring. The study of this invariant became more popular after Kirby applied their result to modules defined by a cokernel matrix between free modules in [21]. Following work in which Gaffney and others found applications of this invariant, Kleiman and Thorup provided a comprehensive presentation of Buchsbaum-Rim multiplicity in the context of algebraic geometry [23].

Another approach to defining the same invariant was more recently provided by Simis, Ulrich, Vasconcelos [36]. The authors define a multiplicity criterion characterizing when a standard graded algebra is integrally closed within a containing standard graded algebra. Given a submodule M of a free module, we may define a Rees algebra of M similar the Rees algebra of an ideal. When applied to these algebras, this relative multiplicity specializes to the existence of Buchsbaum-Rim multiplicity. Viewing modules as algebras significantly simplifies the presentation of this invariant and provides an alternative heuristic for calculating the multiplicity. This is the approach my studies have followed.

Let R ring and let $M \subset R^r$ be an R-module. The symmetric algebra of R^r is $\mathscr{S} := R[x_1, \ldots, x_r]$, which has the standard grading according to the total degree of monomials. Note that $\mathscr{S}_1 \cong R^r$ and $\mathscr{S}_n \cong R^{\binom{n+r-1}{r-1}}$ as R-modules. We define the Rees algebra, $\mathscr{R}(M)$, of a module, M, as a subalgebra of $\mathscr{S}$ using the natural inclusion of $M \subset R^r$. For each $m \in M$, the image of m in R^r is an r-tuple $(m_1, \ldots, m_r)$.

Definition 1.28 ([17], 16.2.1)**.** The **Rees algebra** of the module M is defined to be the subalgebra of $\mathscr{S}$ which is generated in degree 1 over R by $\{\sum_{i=1}^{r} m_i x_i | (m_1, \ldots, m_r) \in im(M) \subset R^r\}$.

One will note that the inclusion map of $M \subset R^r$ need not be unique. Indeed this definition is ambiguous in general, though all rings produced this way are equal up to a nilpotent ideal. For a proof of this fact and a more general definition of the Rees algebra, see [10].

In parallel to the ideal case, we identify the graded pieces of $\mathscr{R}(M)$ with 'powers' of M. For example, $M_2 := \mathscr{R}(M)_2$ is identified with a submodule of $R^{\binom{r+1}{2}}$, with each copy of R corresponding to a monomial in the r variables of total degree 2. In this way M_k is the k^{th} power of M_1 within the symmetric algebra. These are formally known as the torsion-free symmetric powers of a module, though I will continue to refer to them simply as powers.

Example 1.29. Let $R = k[x,y]$ and let $M \subset R^2$ be generated by $\begin{bmatrix} xy \\ y^2 \end{bmatrix}, \begin{bmatrix} x^3 \\ y \end{bmatrix}$. If $\mathscr{S} = R[t,s]$, then $\mathscr{R}(M) := R[xyt + y^2 s, x^3 t + ys]$. Now $M_2 = \mathscr{R}(M)_2$ is generated as an R-module by the following polynomials:

$x^2y^2t^2 + 2xy^3ts + y^4s^2$

$x^4yt^2 + xy^2ts + x^3y^2ts + xy^2s^2$

$x^6t^2 + 2x^3yts + y^2s^2$

We may fix an order on the monomials t^2, ts, s^2 and write M_2 as a submodule of R^3 with the generators $\begin{bmatrix} x^2y^2 \\ 2xy^3 \\ y^4 \end{bmatrix}, \begin{bmatrix} x^4y \\ xy^2+x^3y^2 \\ y^3 \end{bmatrix}, \begin{bmatrix} x^6 \\ 2x^3y \\ y^2 \end{bmatrix}$.

We may now define the relative multiplicity of algebras and take the Buchsbaum-Rim multiplicity as a special case. I follow the presentation in [17], section 16.5.

Let $A \subset B$ be standard graded algebras over a local Noetherian ring R. By abuse of notation, we will let A_1 denote the ideal in B generated by the image of A_1 under the given inclusion. Let G be the associated graded ring of A_1 in B. Note that $G := \frac{B[A_1t]}{A_1B[A_1t]}$ has a natural bigrading, as B is graded and $B[t]$ has a grading with respect to t. Denote this grading by $G_{(1,0)} = B_1$ while $G_{(0,1)} = A_1t$. Define a single grade on G by $G_n = \oplus_{i+j=n} G_{(i,j)}$.

Theorem 1.30 (Simis, Ulrich, Vasconcelos, [36])**.** *Let $A \subset B$ be a homogeneous inclusion of standard graded algebras over a local ring R. Let d denote $\dim(B)$. Suppose that $\lambda_R(B_1/A_1) < \infty$. Fix $t > 0$.*

(i) For every $n \geq t-1$, $\lambda_R(B_n/A_{n-t+1}B_{t-1}) = \lambda((B_tG)_n)$

(ii) For $n \gg 0$, $\lambda(B_n/A_{n-t+1}B_{t-1})$ is given by a polynomial $p_t(n)$ of degree

$\dim(B_tG) - 1 = \dim(G/(0:_G B_tG)) - 1 \leq \dim G - 1 = \dim B - 1 = d - 1$

(iii) We may write the polynomial of (ii) as

$$p_t(n) = \frac{e_t(A,B)}{(d-1)!} n^{d-1} + O(n^{d-1})$$

with $e_t(A,B) = e(B_tG)$ if $\dim(G/(0:_G B_tG)) = d$ and $e_t(A,B) = 0$ if $\dim(G/(0:_G B_tG)) < d$.

(iv) For $t \gg 0$, $e_t(A,B) = e(G/(0:_G B_1G^\infty))$ or $= 0$. Thus we may define $e_\infty(A,B)$.

(v) If B is integral over A, then $e_\infty(A,B) = 0$

(vi) If B is equidimensional, universally catenary and $e_\infty(A,B) = 0$, then B is integral over A.

In order to define Buchsbaum-Rim multiplicity, we are interested in the lengths $\lambda(B_n/A_n)$, which appear in the above theorem for $t = 1$. Thus we will prove the more limited case shown in the next proposition.

Proposition 1.31. *Let $A \subset B$ graded algebras over R as above. Again, let G be considered with the $\mathbb{N}$-grading described. Let $d = \dim(B)$ and suppose $\lambda_R(B_1/A_1) < \infty$.*

(i) $\lambda_R(B_n/A_n) = \lambda_R((B_1G)_n)$

(ii) *For $n \gg 0$, $\lambda_R(B_n/A_n)$ is a polynomial function $P(n)$ of degree* $\dim(B_1G) - 1 \leq d - 1$

(iii) *We may write the polynomial $P(n) = \frac{e(A,B)}{(d-1)!}n^{d-1} + O(n^{d-2})$. If* $\dim(G/(0 :_G B_1G)) = d$*, then* $e(A,B) = e(B_1G)$ *and if* $\dim(G/(0 :_G B_1G)) < d$*, then* $e(A,B) = 0$.

Proof. Consider $G_n \cong \frac{B_n}{A_1B_{n-1}} \oplus \frac{A_1B_{n-1}}{A_2B_{n-2}} \oplus \ldots \oplus \frac{A_{n-1}B_1}{A_n} \oplus A_n$. Since B_1G_{n-1} is a subset of G_n, it has the form $\oplus_{i=0}^{n-1} \frac{A_iB_{n-i}}{A_{i+1}B_{n-i-1}}$. Length is additive, so when we take $\lambda((B_1G)_n)$ we get a telescoping sum. This yields $\lambda((B_1G)_n) = \lambda(\frac{B_n}{A_n})$.

Next, B_1G is a graded module over $G/(0 :_G B_1G)$. Since $(0 :_G B_1G) \cap R = (A_1 :_R B_1)$ and $\lambda(B_1/A_1) < \infty$, $(G/(0 :_G B_1G))_0$ is an Artinian ring. By Theorem 1.5, the graded components of $(B_1G)_n$ have finite length and are given by a polynomial $P(n)$ of degree $\dim(G/(0 :_G B_1G)) - 1$. Therefore, $\dim(G/(0 :_G B_1G)) \leq \dim G = \dim(B) = d$. $\square$

This notation allows us to define Buchsbaum-Rim multiplicity.

Definition 1.32 ([17], 16.5.5)**.** Let $M \subset N \subset R^r$ be R-modules with $\lambda(N/M) < \infty$. Following the notation above, let $A := \mathscr{R}(M)$ and $B := \mathscr{R}(N)$, each of which lie in $\mathscr{S} = R[x_1, \ldots, x_r]$. The **relative Buchsbaum-Rim multiplicity** of M and N, denoted $br(M,N)$, is defined to be $e(B_1G)$. The **Buchsbaum-Rim multiplicity** of the module $M \subset F = R^r$, denoted $br(M)$, is the relative Buchsbaum-Rim multiplicity $br(M,F)$.

We give one application of the Buchsbaum-Rim multiplicity. Many of the results on the integral closure of ideals may be extended to more general modules. As we lack a multiplication between module elements, an equation of integral dependence between elements of a module is impossible. However, a definition of integral closure and reduction of modules may be stated in terms of valuation rings, similar to equation 1.2 on page 13. In this definition $\kappa(P)$ for a prime ideal P denotes the field $(R/P)_P$.

Definition 1.33 ([17], 16.1.3)**.** Let $N \subset M$ be modules over a ring R. We say N is a **reduction** of M if for all P, a minimal prime of R, and all $\kappa(P)$-valuation rings V containing R/P, the image of M in $M \otimes \kappa(P)$ is contained in the V-span of the image of N.

Let $N \subset M \subset F$ be finitely generated R-modules with F free. Using the definition of Rees algebras given above and working with 'nice' objects (that is, when R is a domain or M, N have rank), N is a reduction of M exactly when $\mathscr{R}(M)$ is integral over $\mathscr{R}(N)$.

Theorem 1.34 ([17], Corollary 16.5.7)**.** *Let $(R, \mathfrak{m})$ be a formally equidimensional local Noetherian ring with dimension $d > 0$, and let $N \subset M \subset F = R^r$ be R-modules with $\lambda(F/N) < \infty$. Then N is a reduction of M in F if and only if $br(M,N) = 0$ if and only if $br(M) = br(N)$.*

1.3 j and ε Multiplicities

Achilles and Manaresi [1] first proposed a multiplicity of ideals which are not $\mathfrak{m}$-primary, known as the j-multiplicity. Its properties were studied by various authors and led to development of ε-multiplicity by Katz, Validashti [20] and Ulrich, Validashti [42]. Although the ε-multiplicity has a similar form to the j-multiplicity, the differing action of the local cohomology functor in each case leads to critical differences in the theory.

We continue the development of reductions in order to introduce analytic spread and, in turn, these generalized multiplicities.

Definition 1.35 ([17], 5.15)**.** Given a local ring $(R, \mathfrak{m})$ with ideal I, we may define the **fiber cone** of I as $\mathscr{F}_I := R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \ldots$. The **analytic spread** of I, $\ell(I)$, is defined as $\dim(\mathscr{F}_I)$.

Let G_I be the associated graded algebra of an ideal I. One may note that $\mathscr{F}_I = G_I/\mathfrak{m}G_I$ so that $\ell(I) = \dim(\mathscr{F}_I) \leq \dim(G_I) = \dim(R)$. When equality holds, I is said to have maximal analytic spread.

In Proposition 1.13, we observed that $J \subset I$ is a reduction of ideals if and only if $\mathscr{R}(I)$ is a finite module over $\mathscr{R}(J)$. Because of the relationship between Rees algebras and reductions, we are able to use the fiber cone to study minimal reductions.

Proposition 1.36 ([17], 8.2.4)**.** *Let $(R, \mathfrak{m})$ be a local ring, and let J, I be ideals with $J \subset I^n$, and B the subalgebra of $\mathscr{F}_{I^n}(R)$ generated by $(J + \mathfrak{m}I^n)/\mathfrak{m}I^n$. Then $J \subset I^n$ is a reduction if and only if $\mathscr{F}_I$ is module-finite over B.*

Proof. Since $\mathscr{F}_I$ is module-finite over $\mathscr{F}_{I^n}$, it is enough to prove the case $n = 1$.

Suppose $J \subset I$ is a reduction. By 1.13, $R[Jt] \subset R[It]$ is module-finite. This property is preserved for $R[Jt]/(\mathfrak{m}R[It] \cap R[Jt]) \subset R[It]/\mathfrak{m}R[It]$. But $R[Jt]/(\mathfrak{m}R[It] \cap R[Jt])$ is isomorphic to B and $R[It]/\mathfrak{m}R[It]$ is $\mathscr{F}_I(R)$.

Suppose $\mathscr{F}_I(R)$ is module-finite over B. Let the homogeneous generators of $\mathscr{F}$ lie in degree d or smaller. Then $I^{d+1}/\mathfrak{m}I^{d+1} \subset (J+\mathfrak{m}I)/\mathfrak{m}I)(I^d/\mathfrak{m}I^d)$. Since $I^{d+1} \subset JI^d + \mathfrak{m}I^{d+1}$, Nakayama's lemma demonstrates that $I^{d+1} \subset JI^d$. □

Corollary 1.37 ([17], 8.2.5)**.** *Let R be a Noetherian local ring and let $J \subset I$ be a reduction. The minimal number $\mu(J)$ of generators of J is at least $\ell(I)$.*

We say that $J \subset I$, a reduction, is a **minimal reduction** if there is no ideal K properly contained in J which is a reduction of I. If J is a minimal reduction of an ideal I, it also has no proper reductions of itself.

Proposition 1.38 ([17], 8.3.3)**.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let J be a minimal reduction of I. Then*

(i) $J \cap \mathfrak{m}I = \mathfrak{m}J$

(ii) For any ideal K such that $J \subset K \subset I$, any minimal generating set of J may be extended to a minimal generating set of K.

Proof. Let $L = J \cap \mathfrak{m}I$. Since $\mathfrak{m}J \subset L$, we may set t to be $\lambda_{R/\mathfrak{m}}(J/L)$. Then $J = (x_1,\ldots,x_t)+L$ for some $x_1,\ldots,x_t$ in J. It may be shown ([17], Lemma 8.18) that $(x_1,\ldots,x_t)$ is also a reduction of I. Therefore, if J is a minimal reduction, then $J = (x_1,\ldots,x_t)$. Since t is the minimal number of generators of J, $L \subset \mathfrak{m}J$, giving (i).

For (ii), note that $J \cap \mathfrak{m}K = \mathfrak{m}J$. The t generators of J are thence part of a minimal generating set of K. □

Tying this result to the properties of the fiber cone, we get a strong characterization of minimal reductions.

Corollary 1.39 ([17], 8.3.5)**.** *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and let $J \subset I$ be a reduction such that $\mu(J) = \ell(I)$.*

(i) J is a minimal reduction of I

(ii) The fiber cone of J is canonically isomorphic to the subalgebra of $\mathscr{F}_I$ generated over k by $(J + \mathfrak{m}I)/\mathfrak{m}I$. This algebra is isomorphic to a polynomial ring in $\ell(I)$ variables over k.

(iii) For any integer a, $J^a \cap \mathfrak{m}I^a = \mathfrak{m}J^a$.

Proof. Let $K \subset J$ be a reduction of I. By 1.38 (ii), a minimal generating set of K may be extended to a minimal generating set of J. By 1.37, K may not have fewer than $\ell(I)$ generators. Hence $\ell(I) = \mu(J)$ gives $K = J$, verifying (i).

Proposition 1.36 tells us that the fiber cone, $\mathscr{F}_I$, is module-finite over B, the subalgebra generated over k by $(J + \mathfrak{m}I)/\mathfrak{m}I$. Hence $\dim B = \ell(I)$. Since J is also generated by $\ell(I)$ elements, B is a polynomial ring over k in $\ell(I)$ variables. There is a natural surjective map $\mathscr{F}_J \to B$. Given that $\mathscr{F}_J$ is also generated by $\ell(I) = \ell(J)$ elements, the kernel of this surjection to $k[x_1, \ldots, x_t]$ must be 0, proving (ii).

To see (iii), note that this isomorphism on the graded pieces of $\mathscr{F}_J \to B$ restricts to maps $J^a/\mathfrak{m}J^a \to (J^a + \mathfrak{m}I^a)/\mathfrak{m}I^a$ with kernel 0. Hence $J^a \cap \mathfrak{m}I^a = \mathfrak{m}J^a$. □

Noetherian local rings always admit minimal reductions. When R has a finite residue field, it may occur that no reduction exists with $\ell(I)$ generators. We may often reduce problems to the case of an infinite residue field (see [17] Section 8.4), in which case every minimal reduction J has $\mu(J) = \ell(I)$.

Proposition 1.40 ([17], 8.3.7)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $I \subset R$ an ideal, $\ell(I) = t$, the analytic spread of I. If k is infinite, then every reduction of I contains a reduction generated by t elements.*

Proof. Let $\mathscr{F}_I$ be the fiber cone of I. Let $J \subset I$ be a reduction. If B is the subalgebra of $\mathscr{F}_I$ generated by $(J + \mathfrak{m}I)/\mathfrak{m}I$, then by 1.36 $\mathscr{F}_I$ is finite over B. Using Noether Normalization, there exist $x'_1, \ldots, x'_t \in B_1$ such that B is module-finite over $A = k[x'_1, \ldots, x'_t]$. (Note that if k is not infinite, we may not guarantee $x'_1,\ldots,x'_t$ lie in degree 1 of B.) Hence $\mathscr{F}_I$ is module-finite over A. Choose $x_i \in J$ in the class of x'_i and let $K=(x_1,\ldots,x_t)R$. Then $K \subset J$ and K is a reduction of I by 1.36. $\square$

Using this proposition, we may establish another bound on $\ell(I)$. Recall that $\mathscr{F}_I$ is a standard graded k-algebra generated by $I/\mathfrak{m}I$. Thus $\dim \mathscr{F}_I \leq \mu(I)$, the minimum size of a generating set of I. Now let J be a minimal reduction of I with $\mu(J)=\ell(I)$. Reductions preserve the height of an ideal, so $ht(I)=ht(J)\leq \mu(J)=\ell(I)$. Hence for any I, $ht(I)\leq \ell(I)\leq \dim(R)$. As a special case of this, we note that if I is $\mathfrak{m}$-primary, then $ht(L)\leq \ell(L)\leq \dim(R)=ht(L)$ and I has maximal analytic spread.

As a final preliminary to j-multiplicity, we introduce an important functor. For an ideal $J\subset R$, define a functor on R-modules, H^0_J, by $H^0_J(M):=\{z\in M | J^n\cdot z=0 \text{ for some } n\}$. This is a left exact functor so when H^0_J is applied to a short exact sequence of R-modules, it produces a long exact sequence of cohomologies. For this reason, $H^0_J(-)$ is known as the **zeroth local cohomology functor of** J. For a local ring $(R,\mathfrak{m})$, we henceforth denote $H^0_{\mathfrak{m}}(-)$ by $\Gamma(-)$.

The j-multiplicity of [1] may be defined by applying Γ to ideal quotients of the form I^{n-1}/I^n. Since I^{n-1} is finitely generated, there is a fixed power of $\mathfrak{m}$ for each n with $\mathfrak{m}^a\Gamma(I^{n-1}/I^n)=0$. In fact, since the associated graded ring of I, G_I, is finitely generated over R/I, there is a fixed power $\mathfrak{m}^a$ which annihilates $\Gamma(G_I)$. That is, there exists $a\in\mathbb{N}$ such that $\mathfrak{m}^a\Gamma(I^{n-1}/I^n)=0$ for all n. The graded pieces of $\Gamma(G_I)$ have finite length which is eventually given by a polynomial.

Definition 1.41. Let R be a ring of dimension d and let $I\subset R$ be an ideal. The ***j*-multiplicity of** I is defined as

$$j(I):=\lim_{n\to\infty}\frac{(d-1)!}{n^{d-1}}\lambda(\Gamma(I^{n-1}/I^n))$$

The annihilator of $\Gamma(G_I)$ contains $\mathfrak{m}^aG_I$, so the dimension of $\Gamma(G_I)$ is at most the Krull dimension of $\mathscr{F}_I=G_I/\mathfrak{m}G_I$. If $\dim(\mathscr{F}_I)<\dim(R)$, then $\lambda(\Gamma(I^{n-1}/I^n))$ is eventually given by a polynomial of degree less than $d-1$ and $\lim_{n\to\infty}\frac{\lambda(\Gamma(I^{n-1}/I^n))}{n^{d-1}}=0$. In fact, $j(I)$ is nonzero exactly when $\ell(I)=\dim(R)$.

It should be noted that if I is an $\mathfrak{m}$-primary ideal, then $\lambda(I^{n-1}/I^n)$ is finite. In this case

$\Gamma(I^{n-1}/I^n) = I^{n-1}/I^n$ and the j-multiplicity corresponds to the Hilbert-Samuel multiplicity.

Many of the useful properties of Hilbert-Samuel multiplicity are shared by the j-multiplicity. Geometrically, the j-multiplicity corresponds to an intersection of varieties. Algebraically, it characterizes ideal reductions. We have a formula given by Katz and Validashti relating the Rees valuations of an ideal to its j-multiplicity.

Theorem 1.42 ([20], Theorem 3.9)**.** *Let $(R, \mathfrak{m})$ be a local ring and $I \subset R$ an ideal with maximal analytic spread. Then for each $v \in \mathcal{V}_{\mathfrak{m}}(I)$, there exists a positive integer $d(I,v)$ such that*

$$j(I) = \sum_{v \in \mathcal{V}_{\mathfrak{m}}(I)} d(I,v) \cdot v(I),$$

where $v(I)$ denotes the least value under v of the elements from I.

In [41], Ulrich and Validashti define a multiplicity which includes j-multiplicity of ideals and Buchsbaum-Rim multiplicity of modules. Let $M \subset F = R^r$.

$$j(M) := \lim_{n \to \infty} \frac{(d+r-1)!}{n^{d+r-1}} \cdot \sum_{i=0}^{n-1} \lambda_r\Big(\Gamma\Big(\frac{M_i F_{n-i}}{M_{i+1} F_{n-i-1}}\Big)\Big)$$

Furthermore, for R not necessarily local, they define $j^i(E,N) := \sum_q j(E_q, N_q)$, where q is taken over all primes $q \in Supp_R(FN/EN)$ with $\dim N_q = i$. Using these definitions, they show:

Theorem 1.43 ([41], 4.3)**.** *Let R be a universally catenary Noetherian ring, $U \subset E$ submodules of a free R-module $F := R^e$, and N a finitely generated locally equidimensional R-module. Assume that $U_p = F_p$ for every minimal prime p in $Supp_R(N)$. The following are equivalent:*

(i) $j(U_q, N_q) = j(E_q, N_q)$ *for every prime ideal q of R.*

(ii) $j(U_q, N_q) \leq j(E_q, N_q)$ *for every prime ideal q of R.*

(iii) $j^i(U,N) = j^i(E,N)$ *for* $1 \leq i \leq \dim(N)$

(iv) $j^i(U,N) \leq j^i(E,N)$ *for* $1 \leq i \leq \dim(N)$

(v) U is a reduction of E on N

When $M = I \subset R$, then $\lambda_r\Big(H^0_{\mathfrak{m}}\Big(\frac{M_iF_{n-i}}{M_{i+1}F_{n-i-1}}\Big)\Big)$ becomes $\lambda\Gamma(I^{n-1}/I^n)$. When $M \subset R^r$ has finite colength, this is the Buchsbaum-Rim multiplicity.

In the case of Hilbert-Samuel multiplicity, additivity of length gives a natural relationship between $\lambda(R/I^n)$ and $\lambda(I^{n-1}/I^n)$. This relationship does not extend to the j-multiplicity, as $\Gamma(-)$ is not additive. In particular, we have the following exact sequences:

$$0 \to I^{n-1}/I^n \to R/I^n \to R/I^{n-1} \to 0$$

$$0 \to H^0_{\mathfrak{m}}(I^{n-1}/I^n) \to H^0_{\mathfrak{m}}(R/I^n) \to H^0_m(R/I^{n-1}) \to H^1_{\mathfrak{m}}(I^{n-1}/I^n) \to \dots$$

We can see by induction on n that $\lambda(H^0_{\mathfrak{m}}(R/I^n)) \leq \sum_{i=1}^n \lambda(H^0_{\mathfrak{m}}(I^{n-i}/I^{n-i+1}))$.

By focusing on $\Gamma(R/I^n)$ instead of the consecutive ideals, Katz and Validashti define a new multiplicity with $\varepsilon(I) \leq j(I)$ ([20], Section 4). The natural limit $\lim_{n\to\infty} \frac{d!\lambda(H^0_{\mathfrak{m}}(R/I^n))}{n^d}$ is not known to converge for a general ring, but the $\limsup$ is sufficient to give a criterion for integral closure.

Ulrich and Validashti [42] give this multiplicity as a relative multiplicity of algebras, obtaining

$$\varepsilon(A|B) := \limsup_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \cdot \lambda_R\Big(\Gamma\Big(\frac{B_n}{A_n}\Big)\Big)$$

Theorem 1.44 ([42], Theorem 2.3)**.** *Let R be a universally catenary ring and let $A \subset B \subset C$ be homogeneous inclusions of standard graded Noetherian R-algebras. Assume C is locally equidimensional and $A_P = C_P$ for every prime P of R that is contracted from an associated prime of C. The following are equivalent:*

(i) *B is integral over A*

(ii) $\varepsilon(A_P|C_P) = \varepsilon(B_P|C_P)$ *for all* $P \in Spec(R)$

(iii) $\varepsilon(A_P|C_P) \leq \varepsilon(B_P|C_P)$ *for all* $P \in Supp_R(B_1/A_1)$ *with* $\dim A_P/PA_P = \dim A_P - 1$

(iv) $\varepsilon(A_P|B_P) = 0$ *for all* $P \in Spec(R)$

(v) $\varepsilon(A_P|B_P) = 0$ *for all* $P \in Supp_R(B_1/A_1)$ *with* $\dim A_P/PA_P = \dim A_P - 1$.

One can see that the equation presenting ε-multiplicity is simpler than that of the generalized

j-multiplicity. Also, by the subadditivity of Γ, $\varepsilon(A|B) \leq j(A|B)$. Hence, the fact that $\ell(I) < d$ implies $j(I) = 0$ means that it also implies $\varepsilon(I) = 0$. Ulrich and Validashti give a non-vanishing condition, [42] Theorem 4.4: If $M \subset F = R^r$ and $\ell(M) = d + r - 1$, then $\varepsilon(M) \neq 0$.

Considerable work was performed by Cutkosky and others on when the lim sup converges as a limit. In [7], it was shown that the limits may converge to irrational numbers, which also demonstrates that the lengths are not eventually polynomial. Quite generally, Cutkosky shows that the limit defining ε always converges for R an analytically unramified local ring of positive dimension; [6], 11.3.

In [18], Jeffries and Montano have given geometric interpretations of the j and ε multiplicities when I is a monomial ideal. The j-multiplicity is given by the normalized volume of the cone of the bounded faces with the origin. The ε-multiplicity is given by the normalized volume of the region bounded above by the bounded faces of the Newton polyhedron and below by the extension of the unbounded faces.

Example 1.45. Let $R = k[x,y]_{(x,y)}$. We may consider the monomials as lattice points in the plane $x^a \cdot y^b \sim (a,b) \in \mathbb{N}^2$. Consider (x^3, x^2y, y^4).

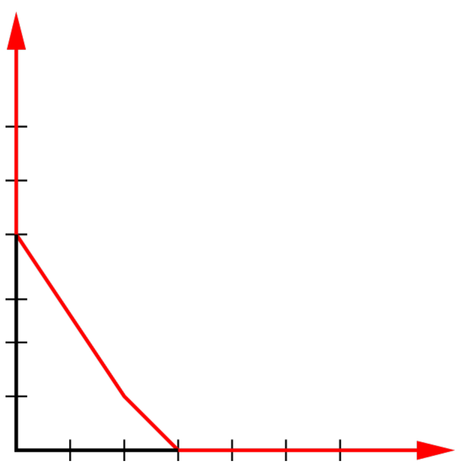


Figure 1.2: $I = (x^3, x^2y, y^4)$

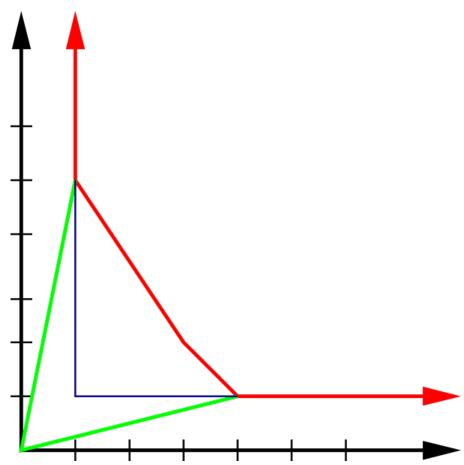


Figure 1.3: $I' = (x^4y, x^3y^2, xy^5)$

In figure 1.2, the area of the region bounded by the Newton polyhedron and the axes is $\frac{11}{2}$, so $e(I) = 11$.

Now consider $I' = (x^4y, x^3y^2, xy^5)$ as shown in figure 1.3. This ideal is not $\mathfrak{m}$-primary, but has maximal analytic spread.

The area of the green cone is 9, which by Theorem 3.2 of [18] gives $j(I') = 18$. The region bounded below by the extension of the unbounded faces is identical to the region found in the first example. By [18] Theorem 5.1, $\varepsilon(I') = 11$.

In Chapter 4, we shall investigate the behavior of $\lambda(\Gamma(R/I^n))$ for ideals I with $\ell(I) < d$. In all cases, this length will be eventually polynomial, though the leading coefficient is no longer useful for a Rees-type criterion. The theory is still in its infancy, and we give examples that indicate interesting behavior.

Chapter 2

Calculating Buchsbaum-Rim Multiplicities

Buchsbaum-Rim multiplicity is known to give useful information on the integral closure of modules and singularities of complete intersections, but the complexity of its calculation is significantly higher than the complexity of a calculation of Hilbert-Samuel multiplicity. Nevertheless, frequent parallels between the two theories encourage us to continue exploring generalizations of results from Hilbert-Samuel theory. This work is initially motivated by the familiar formula $e(I^n) = n^d e(I)$, in which the multiplicity of a power of an ideal I is given in terms of $e(I)$ and $d = \dim(R)$.

As there is no multiplication defined between elements of an arbitrary module, there is not a 'power of a module' which corresponds precisely to the power of an ideal. However, for submodules of free modules, there is a practical analogue which follows naturally from the definition of the Rees algebra. Let $M \subset F = R^r$, a free module of rank r. Recall we defined the Rees algebra of M, $\mathscr{R}(M)$, as the standard $\mathbb{N}$-graded subalgebra of $R[t_1, \ldots, t_r]$ with $\sum_{i=1}^{r} a_i t_i \in \mathscr{R}(M)_1$ if and only if $\begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \in M$. Define $\mathscr{R}(M)_j$ to be the j^{th} torsion-free symmetric power of M, henceforth referred to as the j^{th} power of M. By an abuse of notation, M_j will denote this power; since we do not impose a grading directly on M this notation should not cause confusion. Note that this is a true generalization of ideal powers, since the Rees algebra of I in R is the Rees algebra of I as a submodule of R^1, and $R[It]_j \cong I^j$ as R-modules.

In section 2.1, we define independent mixed Buchsbaum-Rim multiplicities and give a formula for the multiplicity of a direct sum of modules in terms of the independent multiplicities of its summands. Section 2.2 introduces combinatorial functions which produce the multiplicity of the powers of $I \oplus J$ in terms of the mixed Hilbert-Samuel multiplicities of the $\mathfrak{m}$-primary ideals I, J. This is illustrated by direct calculations for the case $M = (I \oplus J)_t$ up to $\dim(R) = 4$ and $t = 4$. Finally, we provide a construction which gives an upper bound on the Buchsbaum-Rim multiplicity of a general module in terms of a direct sum of ideals. This last result sharpens a known bound.

We now introduce the complications which arise from using submodules of free modules, R^r, with $r > 1$. In his thesis [33], Rice suggests a formula for calculating Buchsbaum-Rim multiplicities in his study of symmetric torsion-free powers of a module. We reproduce his calculation of Buchsbaum-Rim multiplicity of a module and its powers in the simplest case.

Example 2.1 ([33], 6)**.** Let $M = \oplus_{i=1}^{r} I$ for an $\mathfrak{m}$-primary ideal I, so that $\mathscr{R}(M) \cong R[Ix_1, \ldots, Ix_r]$.

$$\begin{aligned}
\lambda(F_n/M_n) =& \lambda(\mathscr{R}(F)_n/\mathscr{R}(M)_n) \\
=& \lambda(\oplus_{i=1}^{\binom{n+r-1}{r-1}} R/I^n) \\
=& \sum_{i=1}^{\binom{n+r-1}{r-1}} \lambda(R/I^n) \\
=& \binom{n+r-1}{r-1} \cdot (\frac{e(I)}{d!} n^d + (\textit{lower degree terms})) \\
=& \frac{e(I)}{(r-1)!d!} n^{d+r-1} + (\textit{lower degree terms})
\end{aligned}$$

Hence

$$br(M) = \binom{d+r-1}{r-1} e(I) \tag{2.1}$$

Furthermore, we may apply this formula to calculate $br(M_j)$. Let $t(r) = \binom{j+r-1}{r-1}$.

$$\begin{aligned} M_j &= \oplus_{i=1}^{t(r)} I^j \\ br(M_j) &= \binom{d+t(r)-1}{t(r)-1} e(I^j) \\ &= \binom{d+t(r)-1}{t(r)-1} j^d e(I) \end{aligned}$$

This allows a direct comparison of $br(M)$ with $br(M_j)$.

$$br(M_j) = \frac{\binom{d+t(r)-1}{t(r)-1}}{\binom{d+r-1}{r-1}} j^d br(M)$$

Here, we have a formula for the Buchsbaum-Rim multiplicity of any power of M based on rank of F, dimension of R and $br(M)$. We show in section 2.2 that these three invariants of M alone are insufficient for calculating the multiplicity of symmetric torsion-free powers of M. In fact, the difficulties arise in the next simplest type of module: a direct sum of two different ideals.

We first require a little more background. We will be concerned only with the case of modules $M \subset R^r = F$ which have $\lambda_R(F/M) < \infty$. We wish to analyze $\lim_{n\to\infty} \lambda(F_n/M_n)$.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a local ring. Let $M \subset R^r = F$ be a finitely generated R-module. If $\lambda_R(F/M) < \infty$, then $\lambda_R(F_i/M_i) < \infty$ for all $i \in \mathbb{N}$.*

Proof. Let $\mathscr{R}(M) \subset R[x_1, \ldots, x_r]$ be the Rees algebra of M. In fact, $\mathscr{R}(M)_1$ defines an ideal $I \subset R[x_1, \ldots, x_r]$ which is homogeneously generated in degree 1. Furthermore $(I^n)_n \cong M_n$ as R-modules. Since $F/M \cong (R[x_1, \ldots, x_r]/I)_1$ and $\lambda_R(F/M) < \infty$, $(I :_R (R[x_1, \ldots, x_r])_1)$ is an $\mathfrak{m}$-primary ideal. Moreover, $(I :_R (R[x_1, \ldots, x_r])_1)^i \subset (I^i :_R (R[x_1, \ldots, x_r])_i)$, so the latter ideal is also $\mathfrak{m}$-primary and $\lambda(F_i/M_i) < \infty$. $\square$

In Example 2.1, we observed that for $M \subset R^r$, $\mathscr{R}(M)_n$ was naturally embedded in a free module of rank $\binom{n+r-1}{r-1}$ while $\mathscr{R}(M_t)_n$ was embedded in a free module of rank $\binom{n+j(t)-1}{j(t)-1}$, where $j(t) = \binom{r+t-1}{t-1}$. Our next example gives an indication of how this changes the asymptotic lengths in question.

Example 2.3. Let $M = I \oplus J \subset F = R^2$, which has $\mathscr{R}(M) \cong R[Ix, Jy]$.

$$R[Ix, Jy]_n = I^n x^n + I^{n-1} J x^{n-1} y + \ldots + J^n y^n$$

Hence $M_n \cong \oplus_{i=0}^n I^{n-i} J^i$. Now consider M_2 as a submodule of R^3. We introduce a third variable, z, and define $\mathscr{R}(M_2) = R[I^2 x, IJy, J^2 z]$. Observe that

$$\mathscr{R}(M_2)_2 = I^4 x^2 + I^3 Jxy + I^2 J^2 xz + I^2 J^2 y^2 + IJ^3 yz + J^4 z^2$$

In particular,

$$\begin{aligned} (M_2)_2 &= I^4 \oplus I^3 J \oplus I^2 J^2 \oplus I^2 J^2 \oplus IJ^3 \oplus J^4 \\ &\neq I^4 \oplus I^3 J \oplus I^2 J^2 \oplus IJ^3 \oplus J^4 \qquad = M_4 \end{aligned}$$

and the powers of M_2, after the first, do not coincide with any power of M. When we consider $\mathscr{R}(M_3) = R[I^3 x, I^2 Jy, IJ^2 z, J^3 w]$, we find more contrast:

$$\begin{aligned} (M_3)_4 &= \oplus_{i=0}^{12} I^{12-i} J^i \oplus_{i=2}^{10} I^{12-i} J^i \oplus_{i=3}^{9} I^{12-i} J^i \oplus_{i=4}^{8} I^{12-i} J^i \oplus I^6 J^6 \\ (M_2)_6 &= \oplus_{i=0}^{12} I^{12-i} J^i \oplus_{i=2}^{10} I^{12-i} J^i \oplus_{i=4}^{8} I^{12-i} J^i \oplus I^6 J^6 \\ M_{12} &= \oplus_{i=0}^{12} I^{12-i} J^i \end{aligned}$$

While the powers of I^n are a subsequence of powers of I, powers of a module's powers form disjoint sequences, which makes relating their asymptotic behavior more difficult.

As Example 2.3 indicates, in order to consider asymptotic sequences of such M_n, we must be prepared to consider lengths of the form $\lambda_k(R/I^k J^{n-k})$. The study of mixed Hilbert-Samuel multiplicities has been well developed though it is perhaps less familiar than other results which have been presented.

Let $(R,\mathfrak{m},k)$ be a local ring of Krull dimension d and let $I_1,\ldots,I_t$ be a set of $\mathfrak{m}$-primary ideals. We may consider the length $\lambda(R/\prod I_i^{p_i})$. For $p_1,\ldots,p_t$ sufficiently large, this length is given by a polynomial in t variables $P(p_1,\ldots,p_t)$ of total degree d. Such a polynomial will have $\binom{d+t-1}{t-1}$ terms of highest degree, each of which gives one of the mixed multiplicities of the ideals. Indexing the leading coefficients of P by t-tuples of integers, $\bar{a}=(a_1,\ldots,a_t)$, with $\sum_{i=0}^{t} a_i = d$, we may write

$$P(x_1,\ldots,x_t) = \sum_{a_1+\ldots+a_t=d} \frac{e(\bar{a})}{d!} \binom{d}{a_1,a_2,\ldots,a_{t-1}} \prod x_i^{a_i} + O(d-1)$$

Definition 2.4 ([17], 17.4.3)**.** The coefficients $e(\bar{a})$ as above define the **mixed Hilbert-Samuel multiplicities** of $\{I_1,\ldots,I_t\}$.

This concept has been generalized to modules [22], [23]. Let $\{M^i | M^i \subset R^r \text{ for } 1 \leq i \leq t\}$ be a set of modules of finite colength in R^r. For each i, $\mathscr{R}(M^i) \subset \mathscr{R}(R^r) = R[x_1,\ldots,x_r]$ defines an ideal, I_i, generated by homogeneous elements of degree 1. We consider

$$\lambda\left(R[x_1,\ldots,x_p]_N / \prod_{i=1}^{t} \mathscr{R}(M_i)_{n_i}\right) = \lambda\left((R[x_1,\ldots,x_p]/I_1^{n_1}\cdots I_t^{n_t})_N\right) \text{ where } \sum_{i=1}^{t} n_i = N$$

For $n_1,\ldots,n_t$ each sufficiently large, this length is a polynomial, $P(n_1,\ldots,n_t)$, of total degree $d+p-1$ in t variables. Summing over $\bar{a}\in\mathbb{N}^t$ which have $\sum_{i=1}^{t} a_i = d+p-1$, we may describe the leading form as

$$\sum_{\bar{a}} \frac{e_{\bar{a}}(M^1,\ldots,M^t)}{(d+p-1)!} \binom{d+p-1}{a_1,\ldots,a_{t-1}} \prod_i x_i^{a_i}$$

These $e_{\bar{a}}$ are known as the **mixed Buchsbaum-Rim multiplicities** of $\{M^i\}$ in, for example, [23]. One may see by considering $I_1x,\ldots,I_tx \subset R[x]$ that this is a generalization of Hilbert-Samuel mixed multiplicities. Unfortunately, these constants are not suitable for our purposes in Section 2.1. We give an alternative generalization of the mixed Hilbert-Samuel multiplicities which we call independent mixed Buchsbaum-Rim multiplicities following Theorem 2.12.

The work in this chapter relies on direct calculations of the leading terms of polynomials. We have many expressions which represent polynomials and we wish to proceed from one expression to another based not on whether the polynomials are equal, but on whether the polynomials have the same leading form. Therefore, we establish the following notation for the remainder of the chapter.

Definition 2.5. Define $\approx$ to be the equivalence relation between polynomials in the same variables defined by $P \approx Q$ if P and Q have the same degree and the leading forms of P and Q are equal.

Example 2.6. Let $R = \mathbb{Q}[x,y]$. Then $3x^2+5xy+2y^2 \approx 3x^2+5xy+2y^2-7x+5y-1$.

2.1 Multiplicity of Direct Sums

As we have seen, the simplest nontrivial submodules of R^r are formed by direct sums of ideals; that is, direct sums of submodules of R^1. Direct sums form a natural means of decomposing complicated modules; moreover, the powers of a module which is a direct sum may be expressed as a direct sum of symmetric products of its summands. We now seek a formula for the Buchsbaum-Rim multiplicity of a general direct sum in terms of some multiplicities of its summands. We specifically discuss the implications of this result for $M = (I \oplus J)_t$ in Section 2.2.

The work is most clear when we calculate multiplicity for the direct sum of two ideals over a ring of dimension 1. The proof of our main theorem, 2.18, develops along the same lines.

Note that in the following work, $\lambda(-)$ is always $\lambda_R(-)$.

Example 2.7. Let $(R,\mathfrak{m})$ be a local ring with Krull dimension 1 and let $I,J \subset R$ be $\mathfrak{m}$-primary ideals. The polynomial, P, governing the mixed multiplicities of I and J is degree 1 in two variables: $P(x,y) = e(I)x + e(J)y + a$. Here, $e(I)$ and $e(J)$ are the respective Hilbert-Samuel multiplicities and $a \in \mathbb{Q}$. We claim $br(I \oplus J) = e(I) + e(J)$.

The Rees algebra of $M = I \oplus J \subset R^2$ is $R[Ix, Jy]$. As a submodule of R^{n+1}, the n^{th} power of M has the form

$$M_n = I^n \oplus I^{n-1}J \oplus \ldots \oplus IJ^{n-1} \oplus J^n$$

We seek $\lambda(R[x,y]_n/M_n) = \sum_{i=0}^{n} \lambda(R/I^{n-i}J^i)$.

There exist $r, q \in \mathbb{N}$ such that $\lambda(R/I^a J^b) = P(a,b)$ for any $a \geq r$ and $b \geq q$. Since P is degree 1, $\sum_{i=0}^{n} P(n-i,i)$ is a polynomial of degree 2 in n. For $n \gg 0$, consider the difference

$$\sum_{i=0}^{n} P(n-i,i) - \sum_{i=0}^{n} \lambda(R/I^{n-i}J^i) = \sum_{i=0}^{q-1} \left(P(n-i,i) - \lambda(R/I^{n-i}J^i)\right) + \sum_{i=0}^{r-1} \left(P(i,n-i) - \lambda(R/I^i J^{n-i})\right)$$

Given $i \in \mathbb{N}$, there exists $c \in \mathbb{N}$ such that $\lambda(R/I^{n-i}J^i) \leq \lambda(R/I^{n+c})$ by the Artin-Rees lemma. Since $P(n-i,i)$ is linear and $\lambda(R/I^{n-i}J^i) \leq P(n+c,0)$, $\lim_{n\to\infty} \frac{\sum_{i=0}^{q-1}\left(P(n-i,i)-\lambda(R/I^{n-i}J^i)\right)}{n^2} = 0$. A symmetric argument applies for $\lambda(R/I^i J^{n-i})$ so that

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} P(n-i,i) - \sum_{i=0}^{n} \lambda(R/I^{n-i}J^i)}{n^2} = 0 \tag{2.2}$$

Therefore, the Buchsbaum-Rim multiplicity of $I \oplus J$, which is $\lim_{n\to\infty} \frac{2!}{n^2} \sum_{i=0}^{n} \lambda(R/I^{n-i}J^i)$, may be calculated as $\lim_{n\to\infty} \frac{2!}{n^2} \sum_{i=0}^{n} P(n-i,i)$.

$$\begin{aligned}
&\sum_{i=0}^{n} (e(I)(n-i) + e(J)i + a) \\
=&(e(I)n+a)(n+1) + (e(J)-e(I)) \sum_{i=0}^{n} i \\
=&(e(I)n+a)(n+1) + (e(J)-e(I)) \frac{n^2+n}{2} \\
=&\frac{e(I)+e(J)}{2} n^2 + \frac{e(I)+e(J)+2a}{2} n + a
\end{aligned}$$

Thus, we have shown $br(I \oplus J) = e(I) + e(J)$.

In order to proceed further, we need to define the independent mixed Buchsbaum-Rim multiplicities of a collection of modules.

Definition 2.8. Let R be an $\mathbb{N}^t$-graded algebra. We use $\bar{e_i} \in \mathbb{N}^t$ to denote the element with a 1 in the i^{th} position and zeros elsewhere. We will say that R is a **standard t-graded algebra** if it is generated in degrees $\bar{e_1}, \ldots, \bar{e_t}$. We will call a prime ideal $P \subset R$ **relevant** if it does not contain any of $R_{\bar{e}_1}, \ldots, R_{\bar{e}_t}$. A module M over R has **relevant dimension** given by

$$\max\{\dim(R/P) | P \text{ is a relevant prime and } M_P \neq 0\}$$

Theorem 2.9 ([13], Section 4)**.** *Let R be a standard t-graded algebra over R_0, an Artinian ring. If M is a t-graded R-module, then $\lambda_{R_0}(M_{\bar{n}})$ is a polynomial of degree $(\operatorname{rdim}(M) - t)$ for $\bar{n} \gg 0$.*

For an alternative formulation of this kind of result, see [34], 2.1.7.

Lemma 2.10. *Let R be a standard t-graded algebra over R_0, an Artinian ring. For any nonempty subset $A = \{a_1, \ldots, a_s\} \subset \{1, \ldots, t\}$, there exist integers $m_1, \ldots, m_s$ such that $\lambda_{R_0}(R_{\bar{n}})$ is a polynomial in the s variables $n_{a_1}, \ldots, n_{a_s}$ when $n_{a_i} > m_i$ for all $1 \leq i \leq s$ and $n_j = 0$ for $j \notin A$.*

Proof. Let $A \subset \{1, \ldots, t\}$ be as described. Let S be the $\mathbb{N}^s$-graded algebra generated over R_0 by $R_{a_1} \oplus R_{a_2} \oplus \ldots \oplus R_{a_s}$. By definition, S is a standard s-graded algebra over an Artinian ring. Applying Theorem 2.9, there exist integers $m_1, \ldots, m_s$ such that $\lambda(S_{(n_{a_1}, \ldots, n_{a_s})})$ is a polynomial in s variables, $n_{a_1}, \ldots, n_{a_s}$, when $n_{a_i} \geq m_i$ for each i. Moreover, $S_{(n_{a_1}, \ldots, n_{a_s})} \cong R_{\bar{n}}$ as R_0-modules, when $n_i = 0$ for all $i \notin A$. □

The following theorem allows us to define the independent mixed Buchsbaum-Rim polynomial. The argument is inspired by the proof of Theorem 1.30.

Theorem 2.11. *Let $A \subset B$ be an inclusion of t-graded algebras over $A_0 = B_0 = R$, a local ring. Further suppose that A, B are generated in degrees $\bar{e}_1, \ldots, \bar{e}_t$ and that $\lambda_R(B_{\bar{e}_i}/A_{\bar{e}_i}) < \infty$ for all i. Then there exists $\bar{a} \in \mathbb{N}^t$ such that $\lambda_R(B_{\bar{n}}/A_{\bar{n}})$ is equal to a polynomial in t variables, $P(\bar{n})$, for all $\bar{n} > \bar{a}$ (that is, $n_i > a_i$ for all i). Moreover, the degree of P is given by* $\dim(B) - t$.

Proof. Let $I_1, \ldots, I_t$ be the ideals in B generated by $A_{\bar{e}_1}, \ldots, A_{\bar{e}_t}$, respectively. Let $\mathscr{R}$ be the multigraded Rees algebra of these ideals, $B[I_1x_1, \ldots, I_tx_t]$. This ring has an $\mathbb{N}^{2t}$ grading in which the first t indices indicate the natural grading within B and the next t give the grading in terms of $x_1, \ldots, x_t$. These indices will be denoted by $(\bar{n}|\bar{s})$, for $\bar{n}, \bar{s} \in \mathbb{N}^t$. Denote $\prod_{i=1}^t I_i^{s_i}$ and $\prod_{i=1}^t x_i^{s_i}$ by $I^{\bar{s}}$ and $x^{\bar{s}}$, respectively. Note that $\mathscr{R}_{(\bar{n}|\bar{s})} = B_{\bar{n}} I^{\bar{s}} x^{\bar{s}}$.

Consider $G(j) = \mathscr{R}/I_j\mathscr{R}$.

$$G(j)_{(\bar{n}|\bar{s})} = \begin{cases} \frac{B_{\bar{n}}I^{\bar{s}}x^{\bar{s}}}{I_jB_{\bar{n}-\bar{e_j}}I^{\bar{s}}x^{\bar{s}}} & n_j \geq 1 \\ B_{\bar{n}}I^{\bar{s}}x^{\bar{s}} & n_j = 0 \end{cases} \tag{2.3}$$

We provide an alternative $\mathbb{N}^{2t-1}$ grading on $G(j)$. Let $\bar{s}' \in \mathbb{N}^{t-1}$, so that $(\bar{n}|\bar{s}') \in \mathbb{N}^{2t-1}$.

$$G(j)_{(\bar{n}|\bar{s}')} := \oplus_{i=0}^{n_j} G(j)_{(\bar{n}-i\bar{e_j}|s'_1,\ldots,s'_{j-1},i,s'_j,\ldots,s'_{t-1})}$$

Just as in the proof of Proposition 1.31, we observe that

$$(B_{\bar{e_j}}G(j))_{(\bar{n},\bar{s}')} = \oplus_{i=1}^{n_j} \frac{B_{(\bar{n}-(i-1)\bar{e_j})}I_j^{i-1}I^{\bar{s}'}}{B_{(\bar{n}-i\bar{e_j})}I_j^{i}I^{\bar{s}'}}$$

Since $B_{\bar{e_j}}G(j)$ is annihilated by $(A_{\bar{e_j}} :_R B_{\bar{e_j}})$, it is a graded module over the Artinian ring $R/(A_{\bar{e_j}} :_R B_{\bar{e_j}})$. Therefore, Theorem 2.9 demonstrates that the graded piece $(B_{\bar{e_j}}G(j))_{(\bar{n}|\bar{s}')}$ is a polynomial in $2t-1$ variables for $(\bar{n}|\bar{s}')$ sufficiently high.

We want to show $\lambda(B_{\bar{n}}/A_{\bar{n}}) = \lambda((B/I_1^{n_1}\cdots I_t^{n_t})_{\bar{n}})$ is a polynomial. We demonstrate this through the intermediate expression

$$\begin{aligned} \lambda(B_{\bar{n}}/A_{\bar{n}}) = \lambda((B/I_1^{n_1})_{\bar{n}}) + \lambda((I_1^{n_1}/I_1^{n_1}I_2^{n_2})_{\bar{n}}) + \ldots \\ + \lambda((I_1^{n_1}\cdots I_{t-1}^{n_t}/I_1^{n_1}\cdots I_t^{n_t})_{\bar{n}}) \end{aligned}$$

One may see that the j^{th} term of the above sum is given by the appropriate graded piece of $B_{\bar{e_j}}G(j)$ as defined above.

$$\lambda((I_1^{n_1}\cdots I_{j-1}^{n_{j-1}}/I_1^{n_1}\cdots I_j^{n_j})_{\bar{n}}) = \lambda\left((B_{\bar{e_j}}G(j))_{(0,\ldots,0,n_j,n_{j+1},\ldots,n_t|n_1,\ldots,n_{j-1},0,\ldots,0)}\right)$$

By Lemma 2.10, this is eventually polynomial in $n_1,\ldots,n_t$. Since this holds for all j, there exists $\bar{a}_1,\ldots,\bar{a}_t$ such that if $\bar{n} > \bar{a}_i$ for all i, then $\lambda(B_{\bar{n}}/A_{\bar{n}})$ is the sum of t polynomials in the variables $\bar{n}$

and is itself a polynomial in those variables.

By Theorem 2.9, we have the degree of the polynomial given by $\operatorname{rdim}(B_{\bar{e}_j}G(j)) - (2t-1)$. We think of $B_{\bar{e}_j}G(j)$ as $\frac{B_{\bar{e}_j}\mathscr{R}}{I\mathscr{R}}$. Suppose P is a relevant prime of $\mathscr{R}$ containing I. P does not contain $B_{\bar{e}_j}$, so if $aB_{\bar{e}_j} \subset I \subset P$, then $a \in P$. Hence $\operatorname{rdim}(B_{\bar{e}_j}G(j)) = \dim(\mathscr{R}/I\mathscr{R})$.

Consider the following construction of $\mathscr{R}/I\mathscr{R}$ using the associated graded ring of B with respect to I. Note that $\dim(\frac{B[I_jx_j]}{I_j}) = \dim(B)$. Adjoin to $\frac{B[I_jx_j]}{I_j}$ the remaining ideals and variables: $\frac{B[I_jx_j]}{I_jB[I_jx_j]}[I_1x_1,\ldots,I_tx_t]$. This is a Rees algebra in $t-1$ ideals and since $\lambda(B_{\bar{e}_i}/I_i) < \infty$, none of these ideals is contained in a prime, Q, with $\dim(\frac{B[I_jx_j]}{Q+I_jB[I_jx_j]}) = \dim(\frac{B[I_jx_j]}{I_j})$.

Therefore, $\operatorname{rdim}(B_{\bar{e}_j}G(j)) - (2t-1) = \dim(B) + t - 1 - (2t-1) = \dim(B) - t$ gives the degree of the polynomial. $\square$

In defining the independent mixed Buchsbaum-Rim multiplicities and in the proofs which follow, we will find these notations convenient:

- Index modules by superscripts: $M^1,\ldots,M^t$
- Denote the n^{th} torsion-free symmetric power of M^i by M^i_n
- Indicate an element of $\mathbb{Z}^t$ by $(n_1,\ldots,n_t) = \bar{n}$
- Define $|\bar{n}| = \sum_{i=1}^t n_i$
- For $M^1,\ldots,M^t$ subsets of $R^{r_1},\ldots,R^{r_t}$, respectively, the symmetric product $\prod_{i=1}^t M^i_{n_i}$ will be written $M_{\bar{n}}$
- In this same context, $F_{\bar{n}}$ will be used to denote a free module over R of the appropriate rank to contain $M_{\bar{n}}$: $\prod_{i=1}^t \binom{n_i+r_i-1}{r_i-1}$. For example, if $M^1, M^2 \subset R^3$ and $M = M^1 \oplus M^2 \subset R^6$, then $F_{2,3}$ is a free module whose rank is $\binom{2+2}{2}\binom{3+2}{2} = 60$.

Theorem 2.12. *Let $(R,\mathfrak{m})$ be a local ring of Krull dimension d. Let $M^1,\ldots,M^t$ be a collection of modules for which $\lambda_R(R^{r_i}/M^i) < \infty$. Let $F = R^{\sum_{i=1}^t r_i}$. Let the variables of $\mathscr{R}(F)$ be denoted by $x_{(1,1)},\ldots,x_{(1,r_1)},x_{(2,1)},\ldots,x_{(2,r_2)},\ldots,x_{(t,r_t)} = \bar{x}_1,\bar{x}_2,\ldots,\bar{x}_t$. We may define a subalgebra of $\mathscr{R}(F)$ by $R[M^1\bar{x}_1,\ldots,M^t\bar{x}_t]$. Then there exists a polynomial $P(\bar{n}) = \lambda(F_{\bar{n}}/M_{\bar{n}})$ for all $\bar{n} \gg 0$. The degree of P is $d + (\sum_{i=1}^t r_i) - t$.*

Proof. Let $A = R[M^1\bar{x}_1,\ldots,M^t\bar{x}_t]$ and $B = \mathscr{R}(F)$ be t-graded by the total degree in the $\bar{x_1}$ variables through total degree in the $\bar{x}_t$ variables. Then Theorem 2.11 shows that $\lambda(B_{\bar{n}}/A_{\bar{n}})$ is a polynomial for $\bar{n} \gg 0$ of the appropriate degree. $\square$

Definition 2.13. Let P be as in Theorem 2.12. For $\sum_{|\bar{a}|=D} \frac{D!}{n^D} e(\bar{a})$, the leading form of P, we take $e(\bar{a})$ to be the **independent mixed Buchsbaum-Rim multiplicities** of $M^1,\ldots,M^t$.

We briefly consider an alternative method of verifying the degree, which sheds more light on the leading form of the polynomial. Following [33] 1.3, there is no difficulty in assuming $M^i \subset \mathfrak{m}R^{r_i}$ for each i. Since $\lambda(R^{r_i}/M^i) < \infty$, there exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}_i$ such that $\mathfrak{a}R^{r_i} \subset M^i \subset \mathfrak{m}R^{r_i}$. We may choose $\mathfrak{a}$ such that $\mathfrak{a}F \subset \oplus M^i \subset \mathfrak{m}F$. Now let $|\bar{n}| = N$ and note that

$$\lambda(F_{\bar{n}}/\mathfrak{m}^N F_{\bar{n}}) \leq \lambda(F_{\bar{n}}/M^1_{n_1}\cdots M^t_{n_t}) \leq \lambda(F_{\bar{n}}/\mathfrak{a}^N F_{\bar{n}})$$

Now, the rank of $F_{\bar{n}}$ is given by $\alpha = \prod_{i=1}^t \binom{n_i+r_i-1}{r_i-1}$. This α may be considered a polynomial in t variables, n_i, whose leading form is $\prod_{i=1}^t \frac{n_i^{r_i-1}}{(r_i-1)!}$. It follows that $\lambda(F_{\bar{n}}/\mathfrak{a}^N F_{\bar{n}}) = \alpha\lambda(R/\mathfrak{a}^N)$, where the length is given by a polynomial in N of degree d. The leading form of the polynomial defining $\lambda(F_{\bar{n}}/\mathfrak{a}^N F_{\bar{n}})$ is

$$\frac{e(\mathfrak{a})}{d!}(\sum_{i=1}^t n_i)^d \prod_{i=1}^t \frac{n^{r_i-1}}{(r_i-1)!} \tag{2.4}$$

The polynomial defining $\lambda(F_{\bar{n}}/\mathfrak{m}^N F_{\bar{n}})$ is identical, except that $e(\mathfrak{a})$ is replaced with $e(\mathfrak{m})$. Thus, the polynomial defining $\lambda(F_{\bar{n}}/M^1_{n_1}\cdots M^t_{n_t})$ must have the same degree: $\dim(R) + (\sum_{i=1}^t r_i) - t$. Additionally, equation 2.4 indicates that $e(\bar{a}) = 0$ if there exists i for which $a_i > d + r_i - 1$.

We note that the independent mixed Buchsbaum-Rim multiplicities are again a generalization of the mixed Hilbert Samuel multiplicities of an ideal. If $M^1,\ldots,M^t$ are each submodules of R, then $F = R^t$ under the $\mathbb{N}^t$ grading has $F_{\bar{n}} = R$ for all $\bar{n} \in \mathbb{N}^t$. We provide examples to illustrate the difference between mixed Buchsbaum-Rim multiplicities and the independent mixed Buchsbaum-Rim multiplicities.

Example 2.14. Let $R = k[x]$. Consider $M = \begin{bmatrix} x^3 & x \\ 0 & x^2 \end{bmatrix} \subset R^2$. The Buchsbaum-Rim polynomial of M is $P_M(n) = \frac{5}{2}(n^2+n)$. The mixed Buchsbaum-Rim polynomial of $\{M,M\}$ may be derived from this polynomial by observing $\lambda(R^{a+b+1}/M_aM_b) = \lambda(R^{a+b+1}/M_{a+b}) = \frac{5}{2}((a+b)^2+(a+b))$. Hence, there are 3 mixed Buchsbaum-Rim multiplicity terms which yield $br_0(M,M) = br_1(M,M) = br_2(M,M) = 5$.

The independent mixed Buchsbaum-Rim multiplicities are given by examining the graded pieces of $\mathscr{R}(M \oplus M) \subset R[s,t,u,v]$. The number of monomials in s,t,u,v associated with the $R[s,t,u,v]_{(a,b)}$ is $(a+1)(b+1)$. Thus, $\lambda(R^{(a+1)(b+1)}/M_aM_b) = \frac{5}{2}((a+1)(b+1)(a+b))$. This a polynomial of total degree 3 in a,b and the independent mixed Buchsbaum-Rim multiplicities are $0,5,5,0$.

Example 2.15. Let $R = k[x,y]$, the polynomial ring in two variables over a field. Let $\mathfrak{m}$ be the homogeneous maximal ideal. Consider $M = \begin{bmatrix} x^3 & x^2y & xy^2 & y^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y \end{bmatrix} \subset R^2$ and $N = \begin{bmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{bmatrix} \subset R^2$. One may calculate $br(M) = 13$ and $br(N) = 3$.

The mixed multiplicities of M and N are found by examining $\left(R[s,t]/(\mathfrak{m}^3s,\mathfrak{m}t)^a(\mathfrak{m}s,\mathfrak{m}t)^b\right)_{a+b}$. Using the fact that $\lambda(R/\mathfrak{m}^n) \approx n^2/2$ for $n \gg 0$, one may deduce that the leading form of the mixed BR polynomial is $\frac{13}{6}a^3 + \frac{5}{2}a^2b + \frac{3}{2}ab^2 + \frac{1}{2}b^3$. Therefore, the mixed Buchsbaum-Rim multiplicities are $13,5,3,3$.

The independent mixed multiplicities arise from $(R[s,t,u,v]/(\mathfrak{m}^3s,\mathfrak{m}t)^a(\mathfrak{m}u,\mathfrak{m}v)^b)_{(a,b)}$. This polynomial is degree 4 and the five independent mixed multiplicities are $0,13,8,3,0$.

Example 2.16. Let $R = k[x,y]$. Consider $M = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix} \subset R^2$ and $I = (x^2,y) \subset R$. Since M is a parameter module, it is easy to see that $br(M) = 3$. The Hilbert-Samuel multiplicity of I is 2. Mixed Buchsbaum-Rim multiplicities are not defined in this situation.

The independent mixed Buchsbaum-Rim multiplicities of I and M are given by examining $\lambda(R^{a+1}/M_aI^b)$. The mixed multiplicities associated to a^3, a^2b, ab^2, b^3 are, respectively, $3,2,2,0$.

Recalling the argument of example 2.7, we wish to extend the observation in equation 2.2. We wish to replace $\lambda(F_{\bar{n}}/M_{\bar{n}})$ with the independent mixed Buchsbaum-Rim polynomial $P(\bar{n})$ for all values of $\bar{n}$ with $N = |\bar{n}|$ instead of only for $\bar{n}$ with each n_i sufficiently large. The following lemma says that this replacement will not change the value of the limit by which we calculate Buchsbaum-Rim multiplicity.

Lemma 2.17. *Let K be a field. Let $f : \mathbb{N}^r \to K$ be a nondecreasing function and suppose there exists a polynomial $P(x_1,\ldots,x_t)$ with coefficients in K of total degree d and an integer a such that $f(n_1,\ldots,n_t) = P(n_1,\ldots,n_t)$ when $n_i > a$ for each i. Further suppose that, leaving any proper subset of $\{x_1,\ldots,x_t\}$ fixed, $f(x_1,\ldots,x_t)$ is a polynomial of degree $\leq d$ in the remaining variables when $\sum_i x_i$ is sufficiently large. Then*

$$\lim_{N\to\infty} \frac{\sum_{|\bar{n}|=N} f(\bar{n}) - P(\bar{n})}{N^{d+t-1}} = 0$$

Proof. The total number of elements $\bar{n} \in \mathbb{N}^r$ with $|\bar{n}| = N$ is $\sum_{i_1=0}^{N}\sum_{i_2=0}^{N-i_1}\cdots\sum_{i_{r-1}=0}^{N-\sum_{j<r-1} i_j} 1$ which is on the order of N^{r-1}. Therefore, $\sum_{|\bar{n}|=N} P(\bar{n})$ has degree $d+r-1$ as a polynomial in N.

There exists a such that if $n_i \geq a$ for all i, then $P(\bar{n}) = f(\bar{n})$. Let $\Lambda_N \subset \mathbb{N}^r$ be the collection of all $\bar{n} \in \mathbb{N}^r$ such that $|\bar{n}| = N$ and $n_i \leq a$. There are $\sum_{i_1}^{a}\sum_{i_2=0}^{N-i_1}\cdots\sum_{i_{r-1}=0}^{N-\sum_{j<r-1} i_j} 1$ elements of Λ_N with $i_1 \leq a$, which is on the order of aN^{r-2}. Multiplying this number by r over-counts $|\Lambda_N|$ by all $\bar{n}$ such that $n_i, n_j \leq a$ for $i \neq j$ and is still a polynomial in N of degree $r-2$.

Since f is nondecreasing and is eventually polynomial in whichever variables are not fixed, each term $f(\bar{n})$ is bounded above by a polynomial of degree at most d: $Q_{\bar{n}}(\bar{n})$.

$$\begin{aligned}
&\sum_{|\bar{n}|=N} (f(\bar{n}) - P(\bar{n})) = \sum_{\Lambda_N} (f(\bar{n}) - P(\bar{n})) \\
\leq &\sum_{\Lambda_N} (Q_{\bar{n}}(\bar{n}) - P(\bar{n})) = \sum_{\Lambda_N} Q_{\bar{n}}(\bar{n}) - \sum_{\Lambda_N} P(\bar{n}) \\
\leq &\max_{\bar{a}\in\Lambda_N} \sum_{\Lambda_N} Q_{\bar{a}}(\bar{n}) - \sum_{\Lambda_N} P(\bar{n})
\end{aligned}$$

This is the difference between two polynomials in N of total degree $d+r-2$, so it is a polynomial of degree at most $d+r-2$. Therefore, we have the desired result

$$\lim_{N\to\infty} \frac{\sum_{|\bar{n}|=N} f(\bar{n}) - P(\bar{n})}{N^{d+r-1}} = 0$$

□

The following is the main theorem of this section.

Theorem 2.18. *Let $(R,\mathfrak{m})$ be a local ring with $\dim(R) = d$. Let $A = \oplus_{i=1}^{t} M^i$ be the direct sum of modules M^i with $M^i \subset R^{r_i}$ and $\lambda(R^{r_i}/M^i) < \infty$ for each i. Let $D = d + (\sum_{i=1}^{t} r_i) - t$. Then we may take $A \subset R^{\sum r_i} = F$ with $\lambda(F/A) < \infty$ and $br(A) = \sum_{|\bar{a}|=D} e(\bar{a})$: the sum of all independent mixed Buchsbaum-Rim multiplicities of the modules $\{M^i\}$.*

In the calculations which follow, we apply a pair of lemmas used by Verma. Verma's calculation, producing $e(I \cdot R[Jt])$ for a pair of $\mathfrak{m}$-primary ideals in R, strongly resembles the calculation of $br(I \oplus J)$ in a ring of arbitrary dimension.

Lemma 2.19 ([43], Lemma 2.7)**.** *For $j = 0,1,2,\dots,d$*

$$\sum_{p=0}^{d-j} \binom{d}{j}\binom{d-j}{p}\frac{(-1)^p(d+1)}{j+p+1} = 1$$

Verma also cites Sivaramakrishnan for:

Lemma 2.20 ([43], Lemma 2.8)**.** *For positive integers n and r,*

$$\sum_{i=1}^{n} i^r = \frac{n^{r+1}}{r+1} + \frac{n^r}{2} + (\textit{lower order terms})$$

Proof. (of Theorem 2.18) By definition, $br(A) = \lim_{N\to\infty} \frac{(D+t-1)!}{N^{D+t-1}}\lambda(F_N/A_N)$. We seek to characterize the leading terms of a polynomial defining $\lambda(F_N/A_N)$ and begin by observing that the powers of a direct sum are the direct sum of the products of its summands.

$$\lambda(F_N/A_N) = \sum_{\bar{n}=N} \lambda(F_{\bar{n}}/M_{\bar{n}})$$

By Theorem 2.12, there exists a polynomial, P, such that $\lambda(F_N/A_N) = \lambda(F_{\bar{n}}/M^1_{n_1}M^2_{n_2}\cdots M^t_{n_t}) = P(n_1,\dots,n_t)$ for $n_1,\dots,n_t$ each sufficiently large. Furthermore, we know that when any proper subset of $\{n_1,\dots,n_t\}$ is fixed, $\lambda(F_{\bar n}/M^1_{n_1}M^2_{n_2}\cdots M^t_{n_t})$ is eventually polynomial in the remaining terms. Therefore, we may apply Lemma 2.17 and write

$$br(A)=\lim_{N\to\infty}\frac{(D+t-1)!}{N^{D+t-1}}\lambda(F_N/A_N)=\lim_{N\to\infty}\frac{(D+t-1)!}{N^{D+t-1}}\sum_{|\bar n|=N}P(\bar n)$$

We now calculate the leading form of $\sum_{|\bar n|=N}P(\bar n)$ from the leading form of P, which is

$$P(x_1,\dots,x_t)\approx\sum_{|\bar a|=D}\frac{e(\bar a)}{D!}\binom{D}{a_1,a_2,\dots,a_{t-1}}\prod_{i=1}^{t}x_i^{a_i}$$

By definition, $e(\bar a)$ are the independent mixed Buchsbaum-Rim multiplicities of $\{M^1,\dots,M^t\}$. We will now be concerned with simplifying

$$\sum_{|\bar n|=N}P(\bar n)\approx\sum_{|\bar n|=N}\sum_{|\bar a|=D}\frac{e(\bar a)}{D}\binom{D}{\bar a}\prod_{i=1}^{t}n_i^{a_i}\tag{2.5}$$

We focus our attention on $n_{t-1}^{a_{t-1}},n_t^{a_t}$, rearranging $\sum_{|\bar n|=N}$ into $\sum_{k=0}^{N}(\sum_{|(n_1,\dots,n_{t-2})|=N-k})\sum_{|(n_{t-1},n_t)|=k}$

$$\sum_{k=0}^{N}(\sum_{|(n_1,\dots,n_{t-2})|=N-k})\sum_{|(n_{t-1},n_t)|=k}\sum_{\bar a}\frac{e(\bar a)}{D!}\binom{D}{\bar a}\prod_{i=1}^{t}n_i^{a_i}$$

$$\sum_{k=0}^{N}(\sum_{|(n_1,\dots,n_{t-2})|=N-k})\sum_{\bar a}\left(\sum_{n_{t-1}=0}^{k}n_{t-1}^{a_{t-1}}n_t^{a_t}\right)\frac{e(\bar a)}{D!}\binom{D}{\bar a}\prod_{i=1}^{t-2}n_i^{a_i}\tag{2.6}$$

Now we may apply a binomial expansion to understand $\sum_{n_{t-1}=0}^{k}n_{t-1}^{a_{t-1}}n_t^{a_t}$:

$$\begin{aligned}\sum_{n_{t-1}=0}^{k}n_{t-1}^{a_{t-1}}n_t^{a_t}&=\sum_{i=0}^{k}i^{a_{t-1}}(k-i)^{a_t}\\&=\sum_{i=0}^{k}i^{a_{t-1}}\sum_{j=0}^{a_t}\binom{a_t}{j}k^{a_t-j}(-i)^j\\&=\sum_{j=0}^{a_t}\binom{a_t}{j}k^{a_t-j}(-1)^j\sum_{i=0}^{k}i^{a_{t-1}+j}\end{aligned}$$

by Lemma 2.20, the leading term of $\sum_{i=0}^{k} i^{(a_{(t-1)}+j)}$ as a polynomial in k is $\frac{k^{(a_{(t-1)}+j+1)}}{a_{(t-1)}+j+1}$. Hence,

$$\sum_{j=0}^{a_t}\binom{a_t}{j}k^{a_t-j}(-1)^j\sum_{i=0}^{k} i^{a_{(t-1)}+j} \approx \sum_{j=0}^{a_t}\binom{a_t}{j}k^{a_t-j}(-1)^j\frac{k^{a_{(t-1)}+j+1}}{a_{(t-1)}+j+1}$$
$$=k^{a_{(t-1)}+a_t+1}\sum_{j=0}^{a_t}\binom{a_t}{j}\frac{(-1)^j}{a_{(t-1)}+j+1}$$

Plugging this back in to 2.6, we have:

$$\sum_{k=0}^{N}\Big(\sum_{|(n_1,\ldots,n_{t-2})|=N-k}\Big)\sum_{\bar{a}}\left(k^{a_{(t-1)}+a_t+1}\sum_{j=0}^{a_t}\binom{a_t}{j}\frac{(-1)^j}{a_{(t-1)}+j+1}\right)\frac{e(\bar{a})}{D!}\binom{D}{\bar{a}}\prod_{i=1}^{t-2}n_i^{a_i} \tag{2.7}$$

The summation $\sum_{j=0}^{a_t}$ resembles the construction in Lemma 2.19. Rearranging the multinomial coefficient will produce the missing terms.

$$\binom{D}{a_1,\ldots,a_{t-1}} = \frac{D!}{\prod_{i=1}^{t} a_i!}$$
$$=(a_{t-1}+a_t+1)\cdot\frac{D!}{(a_{t-1}+a_t+1)!}\cdot\frac{1}{\prod_{i=1}^{t-2} a_t!}\cdot\binom{a_{t-1}+a_t}{a_t}$$
$$=\frac{1}{D+1}\binom{D+1}{a_1,\ldots,a_{t-2}}\binom{a_{t-1}+a_t}{a_t}(a_{t-1}+a_t+1)$$

The last two terms, $\binom{a_{t-1}+a_t}{a_t}(a_{t-1}+a_t+1)$, combine with $\sum_{j=0}^{a_t}\binom{a_t}{j}\frac{(-1)^j}{a_{t-1}+j+1}$ to satisfy the conditions of Lemma 2.19.

$$\sum_{j=0}^{a_t}\binom{a_{t-1}+a_t}{a_t}\binom{a_t}{j}\frac{(-1)^j(a_{t-1}+a_t+1)}{a_{t-1}+j+1} = 1 \tag{2.8}$$

We replace $\binom{D}{\bar{a}}$ by $\frac{1}{D+1}\binom{D+1}{a_1,\ldots,a_{t-2}}$ and omit the terms of equation 2.8, so that 2.7 simplifies to:

$$\sum_{k=0}^{N}\Big(\sum_{|(n_1,\ldots,n_{t-2})|=N-k}\Big)\sum_{\bar{a}}\left(k^{a_{(t-1)}+a_t+1}\frac{e(\bar{a})}{(D+1)!}\binom{D+1}{a_1,\ldots,a_{t-2}}\prod_{i=1}^{t-2}n_i^{a_i}\right)$$

Define $\bar{n}' = (n_1, \ldots, n_{t-2}, n_{t-1} + n_t) \in \mathbb{N}^{t-1}$. This allows us to condense the expression to

$$\sum_{|\bar{n}'|=N} \sum_{|\bar{a}|=D} \frac{e(\bar{a})}{(D+1)!} \binom{D+1}{a_1, \ldots, a_{t-2}} \left(\prod_{i=1}^{t-2} (n_i')^{a_i} \right) (n_{t-1}')^{a_{t-1}+a_t+1} \tag{2.9}$$

This has a similar form to 2.5. The first sum is over $(t-1)$-tuples instead of t-tuples, we have $t-2$ terms in the multinomial coefficient and we have only $t-1$ terms in the product. We may rearrange the sum $\sum_{|\bar{n}'|=N}$ to isolate the last two terms, $(n_{t-2}')^{a_{t-2}}, (n_{t-1}')^{a_{t-1}+a_t+1}$, as in equation 2.6. When we reduce the expression again and set $\bar{n}'' = (n_1, \ldots, n_{t-3}, n_{t-2} + n_{t-1} + n_t)$, we have:

$$\sum_{|\bar{n}''|=N} \sum_{|\bar{a}|=D} \frac{e(\bar{a})}{(D+2)!} \binom{D+2}{a_1, \ldots, a_{t-3}} \prod_{i=1}^{t-3} (n'')_i^{a_i} \cdot (n_{t-2}'')^{a_{t-2}+a_{t-1}+a_t+2}$$

We may iterate this process a total of $t-1$ times, which produces

$$\begin{aligned} &\sum_{|\bar{a}|=D} \frac{e(\bar{a})}{(D+t-1)!} (\sum_{i=1}^{t} n_i)^{(\sum_{j=1}^{t} a_j)+t-1} \\ = &\sum_{|\bar{a}|=D} \frac{e(\bar{a})}{(D+t-1)!} N^{D+t-1} \end{aligned}$$

Thus $br(M) = \sum_{\bar{a}} e(\bar{a})$: the sum of all independent mixed Buchsbaum-Rim multiplicities of the modules $\{M^i\}$. $\square$

Letting $M^1, \ldots, M^t$ each be submodules of R, we immediately recover a result proven by Kirby and Rees, [22], Prop. 4.1.

Corollary 2.21. *Suppose $A = \oplus_{i=1}^{t} I_i$ for $\mathfrak{m}$-primary ideals I_i. Then A has a natural embedding into R^t with finite colength and $br(A) = \sum_{\bar{a}} e(\bar{a})$: the mixed Hilbert-Samuel multiplicities of $\{I_i\}$.*

2.2 Powers of Ideal Sums

In this section, we pursue the consequences of Corollary 2.21 for the evaluation of the Buchsbaum-Rim multiplicity of the powers of the module $I \oplus J$ for $\mathfrak{m}$-primary ideals $I \neq J$. The sum of the mixed Hilbert-Samuel multiplicities of I and J give $br(I \oplus J)$ and the sum of the mixed multiplicities of $I^t, I^{t-1}J, \ldots, J^t$ give $br((I \oplus J)_t)$. We give two approaches to calculating $br((I \oplus J)_t)$. Theorem 2.25 is proven by direct calculation following the pattern of the proof of 2.18 and Theorem 2.27 is proven by enumerating the mixed multiplicities of $I^t, \ldots, J^t$ in terms of the mixed multiplicities of I, J. These calculations lead us to Example 2.26 which shows that for $M \subset R^r$, $br(M)$, r and n are insufficient to describe $br(M_n)$.

The following observation is used to simplify an expression in the direct calculations.

Lemma 2.22. *Let $r_1, \ldots, r_n$ be positive rational numbers such that $\sum_{i=0}^n r_i = 1$. Then for any $j \in \mathbb{N}$, $0 \leq \sum_{i=1}^n r_i i^j \leq n^j$.*

Proof. Since $i^j \leq i^{j+1}$ for all terms i, j in this sum, the maximum value of $\sum_{i=1}^n r_i i^j$ is achieved when $(r_0, \ldots, r_n) = (0, \ldots, 0, 1)$ and the minimum is achieved when $(r_0, \ldots, r_n) = (1, 0, \ldots, 0)$. $\square$

Let I, J be $\mathfrak{m}$-primary ideals in a local ring. Let $M = (I \oplus J)_t \subset R^{t+1}$, so that M_n lives in $R^{\binom{n+t}{t}}$. Since M is the direct sum of submodules of R^1, so is M_n. In particular, M_n is the direct sum of ideals of the form $I^{nt-i}J^i$ for $0 \leq i \leq nt$. There are $\binom{n+t}{t}$ ideals which are direct summands of M_n and $nt+1$ ideals of the form $I^{nt-i}J^i$, so some of these ideals correspond to multiple summands of M_n. We will need to count the number of summands which correspond to each ideal.

Definition 2.23. Let $Q : \mathbb{Z}^3 \to \mathbb{Z}$ be the function which counts the number of monomials of total degree n in $R[I^t x_0, \ldots, J^t x_t]$ associated to $I^{nt-i}J^i$. Define $Q(n,t,i) = 0$ unless $0 < n, 0 < t$ and $0 \leq i \leq nt$. Fixing n, t, the sum of all nonzero terms, $\sum_{i=0}^{nt} Q(n,t,i)$, is $\binom{n+t}{t}$, since this corresponds to the total number of monomials in $R[x_0, \ldots, x_t]_n$.

Focusing on the exponent of J, $Q(n,t,i)$ can be thought of as the integer partitions of i with length at most n (the number of factors) and size at most t (the highest power of J which can be the contribution of a single factor). Note that $Q(n,t,i)$ is symmetric in i; that is $Q(n,t,i) = Q(n,t,nt-i)$ by taking the bijection of partitions $(a_1, \ldots, a_n) \leftrightarrow (t-a_1, \ldots, t-a_n)$.

For fixed n, t, the general q-binomial coefficient $\binom{n+t}{t}_q$ forms a generating function of $Q(n,t,i)$. By definition, $\binom{n+t}{t}_q = \prod_{i=0}^{t} \frac{1-q^{n+t-i}}{1-q^i}$. For integers n, t, this may be simplified to a polynomial in q

of degree nt. Then $Q(n,t,i)$ is the integer coefficient of q^i in this polynomial. Of more direct use for our purposes is the following observation.

Lemma 2.24. $Q(n,t,i) = Q(n,t-1,i) + Q(n-1,t,i-t)$

Proof. We divide the integer partitions of i into two classes. The partitions which contain no part of size t are counted by $Q(n,t-1,i)$. The partitions of i with at least one part of size t are counted by the number of ways to partition the remaining integer $i-t$ using the remaining $n-1$ available parts. $\square$

Let us now examine how this function interacts with our calculation of $br((I \oplus J)_t)$. For this calculation, it is convenient to index the variables of $\mathscr{R}(R^{t+1})$ from 0: $R[x_0,\ldots,x_t]$. We now write $\mathscr{R}(M) = R[I^t x_0, I^{t-1}Jx_1,\ldots,J^t x_t] \subset \mathscr{R}(F)$.

$$\lambda(F_n/M_n) = \sum_{i=0}^{nt} Q(n,t,i)\lambda(R/I^{nt-i}J^i) \tag{2.10}$$

Theorem 2.25. *Let I,J be $\mathfrak{m}$-primary ideals in a local ring, $(R,\mathfrak{m})$, of dimension d. There exist real numbers $q_{t,a,b}$ with $0 < q_{t,a,b} \leq 1$ for all (t,a,b) with $0 \leq b \leq a \leq d$ with such that*

$$br((I \oplus J)_t) = t^d \binom{d+t}{t} \sum_{k=0}^{d} \sum_{j=0}^{k} e_k \binom{d}{k} \binom{k}{j} (-1)^j q_{t,k,j}$$

Proof. Let $M = (I \oplus J)_t$ and let $P(x,y)$ be the mixed Hilbert-Samuel polynomial for I and J. We seek $br(M_n) = \lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \sum_{i=0}^{nt} Q(n,t,i)\lambda(R/I^{nt-i}J^i)$. As in Theorem 2.18, our first step will be to reduce the sum of lengths to a sum of polynomials. Consider

$$\lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \left(\sum_{i=0}^{nt} Q(n,t,i)(\lambda(R/I^{nt-i}J^i) - P(nt-i,i)) \right)$$

By the definition of the mixed Hilbert-Samuel multiplicities of I and J, there exists an a for which $nt - i > a$ and $i > a$ gives $\lambda(R/I^{nt-i}J^i) = P(nt-i,i)$. We abbreviate $\lambda(R/I^{nt-i}J^i) - P(nt-i,i)$ as

$(\lambda - P)$ and write

$$\lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \left(\sum_{i=0}^{a} Q(n,t,i)(\lambda - P) + \sum_{i=nt-a}^{nt} Q(n,t,i)(\lambda - P) \right)$$

Now we use that $\sum_{i=0}^{nt} Q(n,t,i) = \binom{n+t}{t}$ and $Q(n,t,i)$ is weakly increasing for $0 \le i \le \frac{nt}{2}$ to place an upper bound on $Q(n,t,a) = Q(n,t,nt-a)$. We may assume $a+1 \le \frac{nt}{4}$, so that $4a+4 \le nt$ and $nt \le 2(nt-2a-2)$. This last inequality may be interpreted as $nt \le \sum_{i=a+1}^{nt-a-1} 2$. However, $1 \le \sum_{i=a+1}^{nt-a-1} \frac{2}{nt}$ implies that $\frac{Q(n,t,a)}{\binom{n+t}{t}} < \frac{2}{nt}$.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \binom{n+t}{t} \left(\sum_{i=0}^{a} \frac{Q(n,t,i)}{\binom{n+t}{t}} (\lambda - P) + \sum_{i=nt-a}^{nt} \frac{Q(n,t,i)}{\binom{n+t}{t}} (\lambda - P) \right) \\
&\le \lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \binom{n+t}{t} \frac{2}{nt} \left(\sum_{i=0}^{a} (\lambda - P) + \sum_{i=nt-a}^{nt} (\lambda - P) \right) \\
&\approx \lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \frac{n^t}{t!} \frac{2}{nt} \left(\sum_{i=0}^{a} (\lambda - P) + \sum_{i=nt-a}^{nt} (\lambda - P) \right) \\
&= \lim_{n\to\infty} \frac{2(d+t)!}{tn^{d+1} t!} \left(\sum_{i=0}^{a} (\lambda - P) + \sum_{i=nt-a}^{nt} (\lambda - P) \right)
\end{aligned}$$

As in the proof of Lemma 2.17, we may bound each λ by a polynomial of degree d, and take the sums over the polynomial which gives a maximum total value. We have the sum of a finite number of polynomials in degree d divided by n^{d+1}, so the limit is 0, as desired. Now we may calculate $br(M)$ as $\lim_{n\to\infty} \frac{(d+t)!}{n^{d+t}} \sum_{i=0}^{nt} Q(n,t,i) P(nt-i,i)$.

We identify the leading form of $P(nt-i,i)$ as $\sum_{k=0}^{d} \frac{e_k}{d!} \binom{d}{k} (nt-i)^k i^{d-k}$

$$\begin{aligned}
\sum_{i=0}^{nt} Q(n,t,i) P(nt-i,i) &\approx \sum_{i=0}^{nt} Q(n,t,i) \left(\sum_{k=0}^{d} \frac{e_k}{d!} \binom{d}{k} (nt-i)^k i^{d-k} \right) \\
&= \sum_{k=0}^{d} \sum_{i=0}^{nt} Q(n,t,i) \frac{e_k}{d!} \binom{d}{k} (nt-i)^k i^{d-k}
\end{aligned}$$

We apply the binomial expansion to $(nt-i)^k$ to obtain

$$\sum_{k=0}^{d}\sum_{i=0}^{nt} Q(n,t,i)\frac{e_k}{d!}\binom{d}{k}\left(\sum_{j=0}^{k}\binom{k}{j}(nt)^{k-j}(-i)^j\right)i^{d-k} \tag{2.11}$$

$$=\sum_{k=0}^{d}\sum_{j=0}^{k}\left(\sum_{i=0}^{nt} Q(n,t,i)i^{d-k+j}\right)\frac{e_k}{d!}\binom{d}{k}\binom{k}{j}(nt)^{k-j}(-1)^j \tag{2.12}$$

Consider

$$\sum_{i=0}^{nt} Q(n,t,i)i^{d-k+j} = \binom{n+t}{t}\sum_{i=0}^{nt}\frac{Q(n,t,i)}{\binom{n+t}{t}}i^{d-k+j}$$

Since $\frac{Q(n,t,i)}{\binom{n+t}{t}}$ is a sequence of rational numbers which sum to 1, we may apply Lemma 2.22 to assert that $\sum_{i=0}^{nt}\frac{Q(n,t,i)}{\binom{n+t}{t}}i^{d-k+j} \leq (nt)^{d-k+j}$. However, we know that this is bounded below by a polynomial of degree $d-k+j$ in n by the existence of the Buchsbaum-Rim multiplicity, so there exists a real number $0 < q_{t,k,j} < 1$ for which $\lim_{n\to\infty}\sum_{i=0}^{nt}\frac{Q(n,t,i)}{\binom{n+t}{t}}i^{d-k+j} = q_{t,k,j}(nt)^{d-k+j}$.

Returning to equation 2.12, we have

$$\begin{aligned}
br(M) &= \lim_{n\to\infty}\frac{(d+t)!}{n^{d+t}}\sum_{k=0}^{d}\sum_{j=0}^{k}\left(\sum_{i=0}^{nt} Q(n,t,i)i^{d-k+j}\right)\frac{e_k}{d!}\binom{d}{k}\binom{k}{j}(nt)^{k-j}(-1)^j\\
&= \lim_{n\to\infty}\frac{(d+t)!}{n^{d+t}}\sum_{k=0}^{d}\sum_{j=0}^{k}\left(\binom{n+t}{t}q_{t,k,j}(nt)^{d-k+j}\right)\frac{e_k}{d!}\binom{d}{k}\binom{k}{j}(nt)^{k-j}(-1)^j\\
&= \lim_{n\to\infty}\frac{(d+t)!}{n^{d+t}}\sum_{k=0}^{d}\sum_{j=0}^{k}\frac{n^{d+t}t^d}{t!}q_{t,k,j}\frac{e_k}{d!}\binom{d}{k}\binom{k}{j}(-1)^j\\
&= \binom{d+t}{t}t^d\sum_{k=0}^{d}\sum_{j=0}^{k}q_{t,k,j}e_k\binom{d}{k}\binom{k}{j}(-1)^j
\end{aligned}$$

□

The exact value of the constants $q_{t,k,j}$ remains elusive. To illuminate them, we shall calculate expressions for $Q(n,t,i)$ when $t=2,3,4$. This will allow us to write down the multiplicity of several powers of $(I\oplus J)$ and demonstrate that $br(M)$, rank of F and $\dim(R)$ are together insufficient to determine $br(M_n)$.

In the following calculations, we use the relatively common notation $\lfloor a\rfloor$ to indicate the greatest integer z such that $z \leq a$. We use $\lceil a \rceil$ to indicate the least integer z such that $z \geq a$.

The case $t = 1$ is trivial: $Q(n,1,i) = 1$ if $0 \leq i \leq n$ and $= 0$ otherwise.

By Lemma 2.24, $Q(n,2,i) = Q(n,1,i) + Q(n-1,2,i-2) = \sum_{j=0}^{n} Q(n-j,1,i-2j)$. For $i \leq n$, the terms of this sum are 1 until $2j > i$, so the entire sum is $\lfloor \frac{i}{2} \rfloor + 1$. By symmetry, we may say that for $i > n$, $Q(n,2,i) = Q(n,2,2n-i)$ and $2n-i < n$.

$$Q(n,2,i) = \begin{cases} \lfloor \frac{i}{2} \rfloor + 1 & 0 \leq i \leq n \\ \lfloor \frac{2n-i}{2} \rfloor + 1 & n < i \leq 2n \end{cases}$$

We proceed to write down expressions for $Q(n,3,i)$ and $Q(n,4,i)$. Applying Lemma 2.24, $Q(n,3,i) = \sum_{j=0}^{n} Q(n-j,2,i-3j)$ and we may use symmetry to only consider $0 \leq i \leq \frac{3n}{2}$. Every term is zero if $3j > i$ or if $i - 3j > 2(n-j)$, but the latter case is excluded by the condition that $0 \leq i \leq \frac{3n}{2}$.

$$Q(n,3,i) = \begin{cases} \sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} Q(n-j,2,i-3j) & 0 \leq i \leq \frac{3n}{2} \\ \sum_{j=0}^{\lfloor \frac{3n-i}{3} \rfloor} Q(n-j,2,3n-i-3j) & \frac{3n}{2} < i \leq 3n \end{cases}$$

In order to replace $Q(n-j,2,i-3j)$ with our formula in terms of i, we need to separate terms with $n-j < i-3j$ from those with $n-j \geq i-3j$.

$$\begin{array}{ll} \sum_{j=0}^{\lceil \frac{i-n}{2} \rceil - 1} Q(n-j,2,i-3j) + \sum_{j=\max(0,\lceil \frac{i-n}{2} \rceil)}^{\lfloor \frac{i}{3} \rfloor} Q(n-j,2,i-3j) & 0 \leq i \leq \frac{3n}{2} \\ \sum_{j=0}^{\lceil \frac{2n-i}{2} \rceil - 1} Q(n-j,2,3n-i-3j) + \sum_{j=\max(0,\lceil \frac{2n-i}{2} \rceil)}^{\lfloor \frac{3n-i}{3} \rfloor} Q(n-j,2,3n-i-3j)) & \frac{3n}{2} < i \leq 3n \end{array}$$

A simpler formula may be obtained by returning to our consideration of integer partitions. For $0 \leq i \leq \frac{3n}{2}$, we have established that $\sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} (\lfloor \frac{i-3j}{2} \rfloor + 1)$ counts the total number of partitions of i into parts of size at most 3. If we correct this term by subtracting the number of such partitions with more than n parts, we will be counting $Q(n,3,i)$.

Consider a partition of i into $n+a$ parts of size at most 3. Subtracting one from each nonzero part, we have a partition of $\ell = i-n-a$ with parts of size at most 2. This is given by $\sum_{\ell=0}^{i-n-1} \lfloor \frac{\ell}{2} \rfloor + 1$.

$$Q(n,3,i) = \begin{cases} \sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} \left(\lfloor \frac{i-3j}{2} \rfloor + 1 \right) & 0 \le i \le n \\ \sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} \left(\lfloor \frac{i-3j}{2} \rfloor + 1 \right) - \sum_{j=0}^{i-n-1} \left(\lfloor \frac{j}{2} \rfloor + 1 \right) & n < i \le \frac{3n}{2} \\ \sum_{j=0}^{\lfloor \frac{3n-i}{3} \rfloor} \left(\lfloor \frac{3n-i-3j}{2} \rfloor + 1 \right) - \sum_{j=0}^{2n-i-1} \left(\lfloor \frac{j}{2} \rfloor + 1 \right) & \frac{3n}{2} < i < 2n \\ \sum_{j=0}^{\lfloor \frac{3n-i}{3} \rfloor} \left(\lfloor \frac{3n-i-3j}{2} \rfloor + 1 \right) & 2n \le i \le 3n \end{cases}$$

The same reasoning applies to $Q(n,4,i)$.

$$Q(n,4,i) = \begin{cases} \sum_{j=0}^{\lfloor \frac{i}{4} \rfloor} Q(n,3,i-4j) & 0 \le i \le n \\ \sum_{j=0}^{\lfloor \frac{i}{4} \rfloor} Q(2n,3,i-4j) - \sum_{j=0}^{i-n-1} Q(n,3,j) & n < i \le 2n \\ \sum_{j=0}^{\lfloor \frac{4n-i}{4} \rfloor} Q(2n,3,4n-i-4j) - \sum_{j=0}^{3n-i-1} Q(n,3,j) & 2n < i < 3n \\ \sum_{j=0}^{\lfloor \frac{4n-i}{4} \rfloor} Q(n,3,4n-i) & 3n \le i \le 4n \end{cases}$$

For the purposes of calculating multiplicity, we do not need to use the exact values of these functions, but only the coefficient of $n^{d-k+j+t}$ in $\sum_{i=0}^{nt} Q(n,t,i) i^{d-k+j}$. Consider the piecewise polynomial function formed by dropping instances of the floor function from $Q(n,2,i)$:

$$F(n,2,i) := \begin{cases} \frac{i}{2} & 0 \le i \le n \\ \frac{2n-i}{2} & n < i \le 2n \end{cases}$$

One may calculate $Q(n,2,i) - F(n,2,i)$ as 1 or $1/2$ depending on the parity of i, so

$$\lim_{n\to\infty} n^{-d+k-j-2} \sum_{i=0}^{2n} \left(Q(n,2,i) - F(n,2,i)\right) i^{d-k+j} = 0$$

Therefore we may replace $Q(n,2,i)$ with $F(n,2,i)$ in the calculation of $br(M_2)$.

Now consider $\sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} \left(\lfloor \frac{i-3j}{2} \rfloor + 1 \right)$ from $Q(n,3,i)$. We first rewrite this as $\sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} \frac{i-3j}{2}$ and notice that for a given residue of i modulo 3, this defines a polynomial in i. Further, Lemma 2.20 may be applied in each of the three cases to see that $\sum_{j=0}^{\frac{i}{3}} \frac{i-3j}{2} \approx \sum_{j=0}^{\frac{i-1}{3}} \frac{i-3j}{2} \approx \sum_{j=0}^{\frac{i-2}{3}} \frac{i-3j}{2} \approx \frac{i^2}{12}$. In this way, we may replace $Q(n,3,i)$ in the calculation of $br(M_3)$ by

$$F(n,3,i) = \begin{cases} \frac{i^2}{12} & 0 \le i \le n \\ \frac{i^2}{12} - \frac{(i-n)^2}{4} & n < i \le \frac{3n}{2} \\ \frac{(3n-i)^2}{12} - \frac{(2n-i)^2}{4} & \frac{3n}{2} < i \le 2n \\ \frac{(3n-i)^2}{12} & 2n < i \le 3n \end{cases}$$

Substituting each instance of $Q(n,3,i)$ in the formula for $Q(n,4,i)$ with $F(n,3,i)$ and simplifying in the same manner gives us:

$$F(n,4,i) = \begin{cases} \frac{i^3}{144} & 0 \le i \le n \\ \frac{i^3}{144} - \frac{(i-n)^3}{36} & n < i \le 2n \\ \frac{(4n-i)^3}{144} - \frac{(3n-i)^3}{36} & 2n < i \le 3n \\ \frac{(4n-i)^3}{144} & 3n < i \le 4n \end{cases}$$

We now return to the expression 2.12, in which we wish to understand $\sum_{i=0}^{nt} Q(n,t,i) i^{d-k+j}$. To simplify the notation, let $c = d-k+j$.

Setting $t = 2$, we find the summation should be broken up at $i = n$

$$\begin{aligned} &\left(\sum_{i=0}^{n} Q(n,2,i) i^c + \sum_{i=n+1}^{2n} Q(n,2,i) i^c \right) \\ \approx &\frac{1}{2} \left(\sum_{i=0}^{n} i^{c+1} + \sum_{i=0}^{2n} 2ni^c - i^{c+1} - \sum_{i=0}^{n} 2ni^c - i^{c+1} \right) \\ = &\frac{1}{2} \left(\sum_{i=0}^{2n} 2ni^c - i^{c+1} - \sum_{i=1}^{n} 2ni^c - 2i^{c+1} \right) \end{aligned}$$

Repeatedly applying Lemma 2.20, we simplify and obtain $\sum_{i=0}^{2n} Q(n,2,i) i^c = \frac{n^{c+2}}{2!} \frac{2^{c+2}-2}{(c+1)(c+2)}$. Recalling that our mysterious $q_{2,k,j}$ from Theorem 2.25 is $\sum_{i=0}^{2n} Q(n,2,i) i^c = \binom{n+2}{2} q_{2,k,j} (2n)^c$, we have $q_{2,k,j} = \frac{2^{d-k+j+2}-2}{2^{d-k+j}(d-k+j+1)(d-k+j+2)}$. This gives us the following formula.

$$br((I \oplus J)_2) = 2^d \binom{d+2}{2} \sum_{k=0}^{d} \sum_{j=0}^{k} e_k \binom{d}{k} \binom{k}{j} (-1)^j \frac{2^{d-k+j+2}-2}{2^{d-k+j}(d-k+j+1)(d-k+j+2)}$$

Similarly, we calculate $\sum_{i=0}^{3n} Q(n,3,i) i^c$ and obtain

$$\frac{n^{c+3}}{6} \cdot \left(3 - 3 \cdot 2^{c+3} + 3^{c+3}\right) \frac{\alpha}{2}$$

where $\alpha = \frac{1}{c+1} - \frac{2}{c+2} + \frac{1}{c+3}$

For $t = 4$, we have

$$\frac{n^{d+4}}{24} \cdot \left(2 - 3 \cdot 2^{c+4} + 2 \cdot 3^{c+4} - 2 \cdot 4^{c+3}\right) \frac{\beta}{3}$$

where $\beta = \frac{1}{c+1} - \frac{3}{c+2} + \frac{3}{c+3} - \frac{1}{c+4}$.

These expressions yield, respectively,

$$br((I \oplus J)_3) = 3^d \binom{d+3}{3} \sum_{k=0}^{d} \sum_{j=0}^{k} e_k \binom{d}{k} \binom{k}{j} (-1)^j \frac{\alpha}{2 \cdot 3^c} (3 - 3 \cdot 2^{c+3} + 3^{c+3})$$

$$br((I \oplus J)_4) = 4^d \binom{d+4}{4} \sum_{k=0}^{d} \sum_{j=0}^{k} e_k \binom{d}{k} \binom{k}{j} (-1)^j \frac{\beta}{3 \cdot 4^c} (2 - 3 \cdot 2^{c+4} + 2 \cdot 3^{c+4} - 2 \cdot 4^{c+3})$$

We fix d and evaluate the expressions, producing coefficients necessary to calculate $br((I \oplus J)_t)$ from $e_0, \ldots, e_d$, the mixed Hilbert-Samuel multiplicities of I, J.

	d=1	d=2	d=3	d=4
$(I \oplus J)_2$	$3e_0 + 3e_1$	$7e_0 + 10e_1 + 7e_2$	15,25,25,15	31,56,66,56,31
$(I \oplus J)_3$	$6e_0 + 6e_1$	25,40,25	90,180,180,90	301,686,861,686,301
$(I \oplus J)_4$	$10e_0 + 10e_1$	65,110,65	350,770,770,350	1701,4396,5726,4396,1701

Example 2.26. Let $R = k[x,y]$ and consider the ideals $I = (x^3, xy, y^3)$, $J = (x^3, y)$ and $K = (x^2, y^2)$. Let $M = I \oplus J$ and observe that $br(M) = e(I) + e(I|J) + e(J) = 6 + 3 + 3 = 12$.

Since $e(K) = 4$, $br(K \oplus K) = 12$.

However, $br((I \oplus J)_2) = 7(6) + 10(3) + 7(3) = 93$ and $br((K \oplus K)_2) = 7(4) + 10(4) + 7(4) = 96$.

This demonstrates that $\dim(R)$, $\operatorname{rank}(M)$, and $br(M)$ alone are insufficient to determine $br(M_n)$.

The preceding work has been focused on simplifying the direct calculation of the lengths in terms of the mixed Hilbert-Samuel polynomial of I,J. We now turn to the alternative technique of writing the mixed multiplicities of $I^t, I^{t-1}J, \ldots, J^t$ in terms of the mixed multiplicities of I,J. Since $br((I \oplus J)_t)$ is equal to the sum of these former mixed multiplicities by Corollary 2.21, this will give us another expression of $br(I \oplus J)_t$ in terms of the mixed multiplicities of I,J.

Let $\bar{n} \in \mathbb{N}^t$. We recall our use of the notation $\sum_{|\bar{n}|=N}$ to indicate the sum over all elements of $\mathbb{N}^t$ which have $\sum_{i=1}^t n_i = N$. For the following statement and proof, we define $0^0 := 1$.

Theorem 2.27. *Let $(R,\mathfrak{m})$ be a local ring of dimension d and let I,J be $\mathfrak{m}$-primary ideals. Let $e_0(I|J), \ldots, e_d(I|J)$ denote the mixed Hilbert-Samuel multiplicities of I,J in R.*
Then $br((I \oplus J)_t) = \sum_{i=0}^d c_i e_i(I|J)$ for coefficients

$$c_i = \sum_{|\bar{p}|=d} \sum_{\bar{u}} \frac{\binom{d}{\bar{p}-\bar{u},\bar{u}}}{\binom{d}{\bar{p}}} \prod_{j=0}^{t} (t-j)^{p_j - u_j} j^{u_j}$$

where the second sum is over $\{\bar{u} \in \mathbb{N}^{t+1}$ s.t. $|\bar{u}| = i$ and $u_j \leq p_j\}$.

Proof. We start by comparing the lengths which define the mixed multiplicities of $I^t, \ldots, J^t$ with the lengths defining multiplicities of I,J.

$$\lambda\left(R/(I^t)^{w_0}(I^{t-1}J)^{w_1}\ldots(J^t)^{w_t}\right) = \lambda\left(R/(I^{(tw_0+(t-1)w_1+\ldots+w_{t-1})}J^{(w_1+2w_2+\ldots+tw_t)})\right)$$

Let $X_0, \ldots, X_t$ be variables and define polynomials $P(X), Q(X)$ which exchange w_i for X_i in the latter expression for the powers of I and J respectively. That is,

$$P(X) = \sum_{i=0}^{t} (t-i)X_i \qquad Q(X) = \sum_{i=0}^{t} iX_i$$

If $F(z_1,z_2)$ is the mixed Hilbert polynomial of I,J, then $F(P(X),Q(X))$ is the mixed Hilbert polynomial of $I^t, I^{t-1}J, \ldots, J^t$. F is degree d and P,Q are degree 1, so the composition is again degree

d. There are now $t+1$ variables, so the number of leading terms increases accordingly.

$$F(X) := F(P(X), Q(X)) = \tag{2.13}$$

$$\frac{1}{d!}\left(\sum_{i=0}^{d}\binom{d}{i}e_i(I|J)(P(X)^{d-i})(Q(X))^i\right) + \textit{lower order terms} \tag{2.14}$$

This expression is written in terms of the mixed multiplicities e_i of I,J. Rearranging to gather terms associated to the same monomial in $X_0,\ldots,X_t$ will reveal the mixed multiplicities of $I^t,\ldots,J^t$. Let $\bar{p}$ be an exponent vector in $\mathbb{N}^{t+1}$ with $\sum_{i=0}^{t} p_i = d$. Fix i, $0 \le i \le d$ and we will calculate the common coefficient of $e_i(I|J)$ and $X^{\bar{p}}$. We must choose i factors of $X^{\bar{p}}$ from $Q(X)$ and choose the remaining $d-i$ from $P(X)$. Hence we take the sum over all $\bar{u} \in \mathbb{N}^{t+1}$ with $u_j \le p_j$ for all j and $|\bar{u}| = i$.

$$\begin{aligned}
&\frac{1}{d!}\left(\sum_{i=0}^{d}\binom{d}{i}e_i(I|J)(P(X)^{d-i})(Q(X))^i\right)\\
=&\sum_{i=0}^{d}\sum_{|\bar{p}|=d}\left(\frac{1}{d!}\binom{d}{i}e_i(I|J)X^{\bar{p}}\sum_{\bar{u}}C(\bar{p},\bar{u})\prod_{j=0}^{t}(t-j)^{p_j-u_j}j^{u_j}\right)\\
=&\sum_{i=0}^{d}\sum_{|\bar{p}|=d}\sum_{\bar{u}}\frac{1}{d!}\binom{d}{i}C(\bar{p},\bar{u})e_i(I|J)X^{\bar{p}}\prod_{j=0}^{t}(t-j)^{p_j-u_j}j^{u_j}
\end{aligned}$$

We have introduced a factor $C(\bar{p},\bar{u})$ which counts the product of the number of permutations of the factors from $P(X)$ with the number of permutations of factors from $Q(X)$.

$$\begin{aligned}
C(\bar{p},\bar{u}) &= \binom{d-i}{p_1-u_1,\ldots,p_t-u_t}\binom{i}{u_1,\ldots,u_t}\\
\frac{1}{d!}\binom{d}{i}C(\bar{p},\bar{u}) &= \frac{1}{d!}\binom{d}{i}\binom{d-i}{p_1-u_1,\ldots,p_t-u_t}\binom{i}{u_1,\ldots,u_t}\\
&= \frac{1}{d!}\binom{d}{\bar{p}-\bar{u},\bar{u}}
\end{aligned}$$

Where $\binom{d}{\bar{p}-\bar{u},\bar{u}}$ represents the multinomial coefficient over the $2t$ integer terms which add up to d.

This brings us to

$$F(X) \approx \sum_{i=0}^{d} \sum_{|\bar{p}|=d} \sum_{\bar{u}} \frac{1}{d!} \binom{d}{\bar{p}-\bar{u},\bar{u}} e_i(I|J) X^{\bar{p}} \prod_{j=0}^{t} (t-j)^{p_j-u_j} j^{u_j}$$

The coefficient of $X^{\bar{p}}$, which will define one of the mixed multiplicities of $I^t,\ldots,J^t$ is given by

$$\sum_{i=0}^{d} \sum_{\bar{u}} \frac{1}{d!} \binom{d}{\bar{p}-\bar{u},\bar{u}} e_i(I|J) \prod_{j=0}^{t} (t-j)^{p_j-u_j} j^{u_j}$$

The mixed multiplicity of $I^t,\ldots,J^t$ associated with $\bar{p}$ is obtained by multiplying this quantity by $\frac{d!}{\binom{d}{\bar{p}}}$. Therefore, by Theorem 2.18, the Buchsbaum-Rim multiplicity of $(I\oplus J)_t$ is given by

$$\sum_{|\bar{p}|=d} \sum_{i=0}^{d} \sum_{\bar{u}} \frac{\binom{d}{\bar{p}-\bar{u},\bar{u}}}{\binom{d}{\bar{p}}} e_i(I|J) \prod_{j=0}^{t} (t-j)^{p_j-u_j} j^{u_j}$$

$\square$

Example 2.28. Let $d=2$, $t=3$. $P(X)=3x_0+2x_1+x_2$ and $Q(X)=x_1+2x_2+3x_3$ and $F(a,b)$ has a leading form of $\frac{e_0}{2}a^2+e_1ab+\frac{e_2}{2}b^2$. Hence the leading form of $F(P(X),Q(X))$ is given by

$$\frac{1}{2}\left(e_0(3x_0+2x_1+x_2)^2+2e_1(3x_0+2x_1+x_2)(x_1+2x_2+3x_3)+e_2(x_1+2x_2+3x_3)^2\right)$$

To calculate the coefficient of x_1x_2, we use $\bar{p}=(0,1,1,0)$. For $i=0$, $\bar{u}=(0,0,0,0)$ and we take both factors from $P(X)=3x_0+2x_1+x_2$ as in the equation. Note that $C(\bar{p},0)=\binom{2}{1,1}=2$, indicating that we find two copies of $2x_1x_2$ in $P(X)^2$, so that $4x_1x_2e_0$ is one term of the polynomial in parantheses. Continuing in this way, we find $(2e_0+5e_1+2e_2)$ to be the coefficient of x_1x_2 in $F(P(X),Q(X))$. The mixed Hilbert-Samuel multiplicity associated to I^2J, IJ^2 is obtained from this by multiplication with $2!/\binom{2}{1,1}=1$.

One may compare the formula derived in Theorem 2.27 to the coefficients of e_i in Theorem 2.25.

$$t^d\binom{d+t}{t}\binom{d}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j q_{t,i,j} = \sum_{|\bar{p}|=d}\sum_{\bar{u}} \frac{\binom{d}{\bar{p}-\bar{u},\bar{u}}}{\binom{d}{\bar{p}}} \prod_{j=0}^{t}(t-j)^{p_j-u_j} j^{u_j}$$

2.3 A New Upper Bound

Given a module $M \subset R^r$ of finite colength, R^r/M has an $\mathfrak{m}$-primary annihilator $\mathfrak{a}$. In [37] page 642, Simis, Ulrich, Vasconcelos note that $br(M) \leq \binom{n+r-1}{r-1} e(\mathfrak{a})$ is a starting point for estimating these multiplicities which invites improvement. In this section, we provide a refinement of this bound through an estimation of M by a direct sum of ideals.

In fact, the bound estimate above is an estimate by a direct sum of ideals. As in equation 2.1, $\binom{n+r-1}{r-1} e(\mathfrak{a}) = br(\mathfrak{a}R^r) = br(\oplus_{i=1}^r \mathfrak{a})$. It is natural to search for a suitable choice of ideals I_i with $br(M) \leq br(\oplus_{i=1}^r I_i) \leq br(\mathfrak{a}R^r)$. Rather than having every ideal annihilate the entire module F/M, we start by identifying the annihilator of $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Let $g'_1, \dots, g'_r$ be standard basis elements of the free module $F = R^r$ containing M. Let $g_1, \dots, g_r$ be their image under the natural inclusion in F/M. Consider the map $\phi_r : R \to F/M$ given by $\phi_r(a) = ag_r$. As F/M has finite length, $ker(\phi_r)$ is an $\mathfrak{m}$-primary ideal, I_r. Moreover, I_r is the annihilator of the last coordinate of R^r/M, so $\mathfrak{a} \subset I_r$. Let C_{r-1} be defined as the cokernel of ϕ_r so that we have an exact sequence

$$0 \longrightarrow I_r \longrightarrow R \xrightarrow{\phi_r} R^r/M \longrightarrow C_{r-1} \longrightarrow 0$$

Now C_{r-1} has a natural embedding into R^{r-1} and can be seen to have finite length by the exact sequence $0 \to R/I_r \to R^r/M \to C_{r-1} \to 0$. In the same manner, we may define I_{r-1} from C_{r-1} using multiplication by g_{r-1}. More precisely, let $c_{r-1,i}$ be the image of g_i in C_{r-1} for $0 < i < r$. The elements of $\{c_{r-1,i}\}$ generate C_{r-1} as a finite length R-module of the form R^{r-1}/M'. Let $I_{r-1} := ker(\phi_{r-1})$ where $\phi_{r-1} : R \to C_{r-1}$ takes $\phi_{r-1}(a) = ac_{r-1,r-1}$. Continuing until we reach $C_1 = R/M^*$, we find $I_1 := M^*$ is an $\mathfrak{m}$-primary ideal and $C_0 = 0$.

Definition 2.29. Given a finite colength module $M \subset R^r$, a set of $\mathfrak{m}$-primary ideals $\{I_1, \dots, I_r\}$

satisfying exact sequences of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_r & \longrightarrow R & \xrightarrow{\phi_r} & R^r/M & \longrightarrow & C_{r-1} & \longrightarrow 0 \\
0 & \longrightarrow & I_{r-1} & \longrightarrow R & \xrightarrow{\phi_{r-1}} & C_{r-1} & \longrightarrow & C_{r-2} & \longrightarrow 0 \\
\vdots & & & & & & & & \\
0 & \longrightarrow & I_2 & \longrightarrow R & \xrightarrow{\phi_2} & C_2 & \longrightarrow & C_1 & \longrightarrow 0 \\
0 & \longrightarrow & I_1 & \longrightarrow R & \xrightarrow{\phi_1} & C_1 & \longrightarrow & 0 &
\end{array}$$

will be called an **annihilating sequence** of M in R^r.

Example 2.30. Let $R = k[x]$ and let $M \subset R^2$ be generated by $\begin{bmatrix} x^3 & x \\ 0 & x^2 \end{bmatrix}$. We seek $(M :_R \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right])$. One may see that $x^2 \left[\begin{smallmatrix} x \\ x^2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} x^3 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ x^4 \end{smallmatrix}\right] \in M$, so $\ker(\phi_2) = (x^4)$. Then $C_1 = R/M^* \cong \frac{R}{(x^3)+(x)} = R/(x)$ is the cokernel. This gives $I_1 = (x)$ and $I_2 = (x^4)$. Notice that if we exchange the indices so that M is generated by $\begin{bmatrix} 0 & x^2 \\ x^3 & x \end{bmatrix}$, we calculate $I_1' = (x^2)$ and $I_2' = (x^3)$. Hence $\{(x),(x^4)\}$ and $\{(x^2),(x^3)\}$ are both annihilating sequences of M in R^2.

Example 2.31. Let $R = k[x,y]$ and let $M = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. Since $\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \in M$, y annihilates $g_2 \in R^2/M$. Also, $\left[\begin{smallmatrix} 0 \\ x^2 \end{smallmatrix}\right] = x\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] - y\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in M$, so $\phi_2 = (x^2, y)$. Omitting the second row of the matrix, $\phi_1 = (x,y)$. Another annihilating sequence is $(x,y^2),(x,y)$.

Theorem 2.32. *Let $(R,\mathfrak{m})$ be a local ring. Let M be a module of finite colength in $F = R^r$, and let $\{I_1,\ldots,I_r\}$ be an annihilating sequence. Then $\sum_{i=1}^r \lambda(R/I_i) = \lambda(F/M)$ and $br(M) \leq br(\oplus_{i=1}^r I_i)$.*

Proof. By definition of the annihilating sequence, for each $0 < k \leq r$ we have a short exact sequence

$$0 \longrightarrow R/I_k \xrightarrow{\phi_k} C_k \longrightarrow C_{k-1} \longrightarrow 0$$

By additivity of λ, we get $\lambda(F/M) = \lambda(R/I_r) + \lambda(C_{r-1}) = \sum_{i=1}^r \lambda(R/I_i) + \lambda(C_0) = \sum_{i=1}^r \lambda(R/I_i)$.

In order to prove the inequality, we must examine the cokernel modules more closely. It is helpful for this purpose to consider elements of M as homogeneous degree 1 polynomials of $\mathscr{R}(M) \subset R[x_1,\ldots,x_r]$. Denote by $M^{[k]}$, the submodule of R^k formed from M by substituting 0 for each of the last $r-k$ variables. One may see by the definition of the maps ϕ_r that $C_k = \frac{R^k}{M^{[k]}}$. Hence $a \in I_k$ if and only if $\exists s_{k+1},\ldots,s_r \in R$ with $ag'_k + \sum_{i=k+1}^r s_i g'_i \in M$.

We establish the lexicographic ordering on monomials in the variables $x_1,\ldots,x_r$ which have the same total degree. That is, let $\sum_{i=1}^r p_i = \sum_{i=1}^r q_i$. We say $\prod_{i=1}^r x_i^{p_i} > \prod_{i=1}^r x_i^{q_i}$ if and only if $p_1 = q_1, p_2 = q_2, \ldots, p_{k-1} = q_{k-1}$ and $p_k > q_k$ for some $1 \leq k \leq r$. With this ordering, we may form an annihilating sequence $J_{\bar{n}}$ of M_N in F_N. The ideals $J_{\bar{n}}$ are indexed by r-tuples with $\sum_{i=1}^r n_i = N$, so that $J_{\bar{n}}$ corresponds to the monomial $\prod_{i=1}^r x_i^{n_i}$ in $\mathscr{R}(F)_N$. We claim $\prod I_i^{n_i} \subset J_{\bar{n}}$.

Let $a \in \prod_{i=1}^r I_i^{n_i}$ so that $a = \prod_{k=1}^N a_k$. For each $a_k \in I_i$, there exists a polynomial in $\mathfrak{R}(M)_1$ of the form $f_k = a_k g_i + \sum_{j=i+1}^r s_j g_j$. For $1 \leq i \leq r$, $a_k g_i$ is the least term in exactly n_i of the polynomials f_k. Hence $\prod_{k=1}^N f_k$ is an element of $\mathfrak{R}(M)_N$ whose least term is $a\prod_{i=1}^r g_i^{n_i}$. All the later monomials will be truncated at the formation of $J_{\bar{s}}$ for some $\bar{s} > \bar{n}$, so a will be added to $J_{\bar{n}}$. Thus, $\prod I_i^{n_i} \subset J_{\bar{n}}$.

Since $J_{\bar{n}}$ is an annihilating sequence for $M_N \subset F_N$, $\lambda(F_N/M_N) = \sum_{|\bar{n}|=N} \lambda(R/J_{\bar{n}})$. Now we apply $\prod I_i^{n_i} \subset J_{\bar{n}}$ to find

$$\begin{aligned}\lambda(F_N/M_N) &= \sum_{|\bar{n}|=N} \lambda(R/J_{\bar{n}}) \\ &\leq \sum_{|\bar{n}|=N} \lambda(R/(\prod I_i^{n_i})) = \lambda(F_N/(\oplus I_i)_N)\end{aligned}$$

Since this inequality holds for each N, it holds for the limit

$$\lim_{N\to\infty} \frac{(d+r-1)!}{N^{d+r-1}} \lambda(F_N/M_N) \leq \lim_{N\to\infty} \frac{(d+r-1)!}{N^{d+r-1}} \lambda(F_N/(\oplus I_i)_N)$$

Therefore $br(M) \leq br(\oplus_{i=1}^r I_i)$. □

Corollary 2.33. *Let $(R,\mathfrak{m})$ be a local ring. Let $M \subset R^r$ and let $I_1,\ldots,I_r$ be an annihilating sequence of M. Then $br(M) \leq \sum_{|\bar{a}|=\dim(R)} e_{\bar{a}}(I_1,\ldots,I_r)$, where $e_{\bar{a}}$ are the mixed Hilbert-Samuel multiplicities of $I_1,\ldots,I_r$.*

Proof. Apply Corollary 2.21 to the result of Theorem 2.32. □

It is also worth noting that if $\alpha \in (M :_R R^r)$, then $\alpha \in I_k$ for any ideal in an annihilating sequence. Thus $\lambda(R/I_k) \leq \lambda(R/\mathfrak{a})$ and $br(\oplus_{i=1}^r I_i) \leq br(\oplus_{i=1}^r \mathfrak{a})$. Corollary 2.33 gives a way to calculate estimates of Buchsbaum-Rim multiplicity that are closer than those given by $\mathfrak{a}$ while still working with Hilbert-Samuel multiplicities.

Example 2.30 concerns a simple enough module to calculate the Buchsbaum-Rim multiplicity directly. Let $R = k[x]$ and let M be given by $\begin{bmatrix} x^3 & x \\ 0 & x^2 \end{bmatrix}$. We directly calculate $br(R^2/M)$ by studying the lengths $\frac{k[x,t,s]_n}{(x^3t, xt+x^2s)^n}$ where $\deg(x) = 0, \deg(t) = \deg(s) = 1$.

$$I^n := (x^3t, xt+x^2s)^n = (x^{3n}t^n, x^{3n-2}t^{n-1} + x^{3n-1}t^{n-1}s, \ldots, x^nt^n + \ldots + x^{2n}s^n)$$

It may be observed that x^{3n} is the smallest power of x that multiplies t^n into I^n. By the second generator, $t^{n-1}s$ is multiplied into linear dependence with a multiple of t^n by (x^{3n-1}). Proceeding termwise, we see that $\lambda\left(\frac{k[x,t,s]_n}{(x^3t, xt+x^2s)^n}\right) = \sum_{i=0}^n 3n - i = 3n(n+1) - \frac{n(n+1)}{2}$, and $br(R^2/M) = 5$.

Now consider the annihilating sequences of M, $\{(x^4), (x)\}$ and $\{(x^2), (x^3)\}$. To calculate $br((x^a) \oplus (x^b))$, we examine $\lambda\left(\frac{k[x,t,s]_n}{(x^at, x^bs)^n}\right)$. Since $(x^at, x^bs)^n = (x^{an}t^n, x^{(a-1)n+b}t^{n-1}s, \ldots, x^{bn}s^b)$, the length is given by $\sum_{i=0}^n a(n-i) + ib = an(n+1) - a\frac{n(n+1)}{2} + b\frac{n(n+1)}{2}$. Twice the leading coefficient is $a+b$.

The two annihilating sequences have $a = 1, b = 4$ and $a = 2, b = 3$, respectively, so that both estimates give the multiplicity exactly. The annihilator of R^2/M is (t^4); using $a = 4, b = 4$ gives an upper bound of 8 on $br(M)$.

Similarly, one may perform the same calculations with Example 2.31. Here, M is a parameter module so its Buchsbaum-Rim multiplicity is known to be 3. The two annihilating sequences each give an upper bound of 4 while the multiplicity of the annihilator, (x^2, xy, y^2), gives an upper bound of 12.

Chapter 3

Multiplicities of Filtrations

We wish to generalize to certain filtrations notions belonging to the theory of multiplicity of ideals. We first define filtrations as a generalization of valuations and of the sequence of powers of an ideal. There is already a great deal of literature on filtrations, from which we highlight Noetherian filtrations, reductions of filtrations and a 'multiplicity' for filtrations of $\mathfrak{m}$-primary ideals. In Section 3.2, we examine the question of whether the Rees valuations of an ideal are Noetherian filtrations. Finally, we extend results on the j-multiplicity of ideals to a j-multiplicity for Noetherian filtrations.

3.1 Introduction

Given a ring R, we say that a map $f : R \to \mathbb{R} \cup \{\infty\}$ is a filtration if it satisfies the following conditions:

(i) $f(1) \geq 0$

(ii) $f(0) = \infty$

(iii) $f(x \cdot y) \geq f(x) + f(y)$

(iv) $f(x+y) \geq \min(f(x), f(y))$

Following [31] 2.1, we say two filtrations f, g are **equivalent** if there exists a constant K with $|f(x) - g(x)| < K$ for all $x \in R$. It follows that every filtration is equivalent to a map $f : R \to \mathbb{Z} \cup \{\infty\}$.

We now assume all filtrations give integer-valued functions.

A filtration naturally gives rise to a sequence of ideals $f_i \subseteq R$, by $f_i = \{r \in R | f(r) \geq i\}$. From the above properties we may see that

(a) $f_0 = R$

(b) $f_{i+1} \subseteq f_i$

(c) $f_i \cdot f_j \subseteq f_{i+j}$

We may also define a filtration as a sequence of ideals on R indexed by the integers which satisfy $(a), (b)$ and (c). These two definitions of a filtration are equivalent, although more recent papers tend to use the latter definition. In the former definition, equality in property (iii) gives a valuation; in the latter definition, equality in property (c) shows that $f_i = f_1^i$. Hence filtrations are generalizations both of ideal powers and of valuations.

The most common examples of filtrations besides powers of ideals are the symbolic powers of primes. One may also consider the filtration $\overline{I^n}$, which is not in general equal to the powers of $\overline{I}$.

Definition 3.1. Let f be a filtration on a ring R. The **Rees ring of the filtration** is defined to be $\mathscr{R}(f) := R[t^{-1}, f_1 t, f_2 t^2, \ldots]$

One may note that if $f_i = f_1^i$, this is simply the extended Rees ring of the ideal f_1. Examples of filtrations that follow will often be defined by a description of this Rees ring. Notice that, given a ring R and a graded extension $\mathscr{R} = R[t^{-1}, x_1 t^{a_1}, x_2 t^{a_2}, \ldots, x_n t^{a_n}]$ where $x_i \in R$ and $a_i \in \mathbb{N}$, the latter ring defines a filtration. Here $f_n = \{x \in R | x t^n \in \mathscr{R}\}$ and $f(x)$ is the highest integer i such that $x \in f_i$.

For any $x \in R$, $f(x) = f(1 \cdot x) \geq f(1) + f(x)$. If $f(1) > 0$, then the filtration degenerates with $f(x) = \infty$ for all $x \in R$. We will assume that $f(1) = 0$ for all future filtrations. The following lemma of Rees gives us some justification for assuming that our filtrations will be homogeneous; that is, $f(x^n) = n f(x)$. The lemma produces a natural homogenization of an arbitrary filtration which preserves the equivalence of filtrations.

Lemma 3.2 ([31], 2.11)**.** *If f is a filtration on a ring R, then $\lim_{n\to\infty} f(x^n)/n$ exists for all $x \in R$ if ∞ is allowed as a limit. If we denote this limit as $\mathbf{f}(x)$, then $\mathbf{f}$ is a homogeneous filtration.*

Furthermore, if h is any homogeneous filtration with $h(x) \geq f(x)$ for all $x \in R$, then $h(x) \geq \mathbf{f}(x)$ for all $x \in R$.

Our work on filtrations in Section 3.2 revolves around Rees valuations. The following definition is a critical component of the proof that Rees valuations are unique.

Definition 3.3. A filtration is known as a **subvaluation** if there exists a finite set of valuations $\{v_1, v_2, \ldots, v_s\}$ such that $f(a) = \min\{v_1(a), \ldots, v_s(a)\}$ for any $a \in R$.

Example 3.4. Let $R = k[x,y]$ and let f be the filtration whose Rees ring is $\mathscr{R}(f) = R[t^{-1}, xyt]$. f is a subvaluation with $f(\alpha) = \min(v_x(\alpha), v_y(\alpha))$ where v_x, v_y are monomial valuations which assign values $v_x(x) = 1, v_x(y) = 0$ and $v_y(x) = 0, v_y(y) = 1$.

We reproduce the proof of another lemma of Rees which employs the concept of f-compatible sets.

Definition 3.5. Let f be a filtration on a ring R. $A \subset R$ is an f**-compatible set** if for any $a, b \in A$, $f(ab) = f(a) + f(b)$.

Note that for any valuation v, $\{x \in R$ such that $v(x) = f(x)\}$ is f-compatible.

Lemma 3.6 ([31], Lemma 2.12)**.** *Let f be a filtration on a ring R. Assume f is a subvaluation and suppose that $\{v_1, \ldots, v_s\}$ is irredundant for the purpose of defining f. That is, for each $1 \leq i \leq s$, there exists $x_i \in R$ such that $f(x_i) \neq \min_{j \neq i}(v_j(x_i))$. Then $\{v_1, \ldots, v_s\}$ is uniquely determined by f.*

Proof. If $f(z) = \infty$, then $v_i(z) = \infty$ for all i. Hence we concern ourselves only with $F \subset R$, the set on which f is finite. If F is empty, then f is trivially a valuation; assume $F \neq \emptyset$.

f-compatible subsets of F form a nonempty set, as every singleton is included by the homogeneity of f. Further, these sets may be ordered by inclusion and we may apply Zorn's Lemma to obtain F as the union of its maximal f-compatible subsets.

Let S_i be the set of all elements $y \in F$ such that $v_i(y) = f(y)$. This set is f-compatible since for all $x, y \in S_i$,

$$f(xy) \geq f(x) + f(y) = v_i(x) + v_i(y) = v_i(xy) \geq f(xy)$$

We claim that $\{S_1,\ldots,S_s\}$ is the set of maximal f-compatible subsets of F. Suppose S is an f-compatible subset which is not contained in any S_i. Then for each i there is an $x_i \in S \setminus S_i$, in which case $f(x_i) < v_i(x_i)$. Then for any given j,

$$\sum_{i=1}^{s} f(x_i) < \sum_{i=1}^{s} v_j(x_i) = v_j(\prod_{i=1}^{s} x_i)$$

Therefore,

$$\sum_{i=1}^{s} f(x_i) < \min_j(v_j(\prod_{i=1}^{s} x_i)) = f(\prod_{i=1}^{s} x_i)$$

But this contradicts the fact that S is f-compatible. We conclude that every f-compatible set is contained in S_i for some i.

Fix $1 \leq i \leq s$. We may choose x_i such that $f(x_i) = v_i(x_i) < v_j(x_i)$ for all $j \neq i$. Hence S_i is not contained in the union of sets $S_j, j \neq i$. Therefore the sets S_i are exactly the maximal f-compatible subsets of F.

Now we show that each S_i uniquely determines the values of v_i. Consider Σ_i to be the subset of F with $a \in \Sigma_i$ if and only if $aS_i \cap S_i \neq \emptyset$. Σ_i contains all elements a with $v_i(a) \neq \infty$. Indeed, let x_i be an element of $S_i \setminus \cup_{j\neq i} S_j$. $v_j(ax_i^m) - v_i(ax_i^m) = v_j(a) - v_i(a) + m(v_j(x_i) - v_i(x_i))$ can be made positive for any a with $v_i(a) > \infty$, so $ax_i^m \in S_i$.

Further, for $a \in \Sigma_i$, $v_i(a)$ is determined by f as follows: since there are $y,z \in S_i$ with $ay = z$, $v_i(a) = v_i(z) - v_i(y) = f(z) - f(y)$. □

Among the most natural classes of filtrations are those arising from a valuation on a ring. Of particular importance are the Rees valuations, which may be defined from a ring by a specific sequence of operations, as in 1.25. Recall the definition of Rees valuations.

Definition 3.7 ([17], 10.1.1)**.** Let R be a ring and I an ideal of R. Suppose that there exist finitely many discrete valuation rings $V_1,\ldots,V_r$ of rank one satisfying the following properties:

(i) For each i there exists a minimal prime ideal P of R such that $R/P \subseteq V_i \subseteq (R/P)_P$. Let $f_i : R \to V_i$ be the natural ring homomorphism.

(ii) For all $n \in \mathbb{N}$, $\overline{I^n} = \cap_{i=1}^{r} f_i^{-1}(I^n V_i)$.

(iii) The set $\{V_1, \ldots, V_r\}$ satisfying (ii) is minimal possible.

The $V_1, \ldots, V_r$ are called Rees valuation rings of I, and the corresponding valuations are called the Rees valuations of I.

By [17] 10.2.2, we know that every ideal in a Noetherian ring has a set of Rees valuations. The Rees valuations associated to an ideal are unique. We prove this using the rational powers of an ideal, as this approach more closely resembles our work in Section 3.2.

Definition 3.8. Let I be an ideal in a ring R and let $\alpha = \frac{p}{q}$ be a positive rational number with $p, q \in \mathbb{N}$. Define $I_\alpha := \{x \in R | x^q \in \overline{I^p}\}$.

We first note that I_α is an ideal. Since $\overline{I^p}$ is an ideal for any p, we have I_α is closed under multiplication from R. That is, if $x^q \in \overline{I^p}$ then $(zx)^q \in \overline{I^p}$. Let $x, y \in I_\alpha$.

$$(x+y)^q = x^q + a_1 x^{q-1} y + a_2 x^{q-2} y^2 + \ldots + y^q$$

But each $x^{q-j} y^j$ lies in $\overline{(x^q, y^q)} \subset \overline{\overline{I^p}} = \overline{I^p}$ so $(x+y)^q \in \overline{I^p}$. Hence I_α is an ideal in R.

Proposition 3.9 ([17], 10.5.2)**.** *$I_\alpha \subset R$ is well-defined as an ideal and the sequence of all rational powers is a subvaluation on R.*

Proof. Let $\alpha = \frac{pi}{qi}$. $x \in I^\alpha$ means $x^{qi} \in \overline{I^{pi}}$, but an equation of integral dependence satisfying this condition also demonstrates that $x^q \in \overline{I^p}$. Since $x^q \in \overline{I^p}$ gives $x^{qi} \in \overline{I^{pi}}$ for all i, I_α is well-defined.

Let $\frac{p}{q} = \alpha \leq \beta = \frac{s}{t}$ be positive rational numbers. Then $x \in I_\beta$ gives $x^t \in \overline{I^s}$ so that $x^{tp} \in \overline{I^{ps}}$. Since $pt \leq qs$, $x^{qs} \in \overline{I^{ps}}$ which shows that $x^q \in \overline{I^p}$ and $x \in I_\alpha$. Thus for all $\alpha \leq \beta$, $I_\beta \subseteq I_\alpha$.

It remains to show that $I_\alpha I_\beta \subset I_{\alpha+\beta}$. Let $x \in I_\alpha$ and $y \in I_\beta$. Since $\alpha + \beta = \frac{pt+sq}{qt}$, we wish to show that $(xy)^{qt} \in \overline{I^{pt+sq}}$. Rearranging the exponents, $(xy)^{qt} = x^{qt} y^{qt} = (x^q)^t (y^t)^q$. By hypothesis $x^q \in \overline{I^p}$ and $y^t \in \overline{I^s}$. Hence $(x^q)^t \in \overline{I^{pt}}$ and $(y^t)^q \in \overline{I^{sq}}$. Since $\overline{I^{pt}} \cdot \overline{I^{sq}} \subset \overline{I^{pt+sq}}$, we have the conclusion. Therefore, the rational powers of I are a filtration.

We now show that this filtration is also a subvaluation. There exist Rees valuations of I, $v_1, \ldots, v_s$ which determine the integral closures of $\{I^n\}$. By definition, $x \in \overline{I^n}$ if and only if

$v_i(x) \geq nv_i(I)$ for all $1 \leq i \leq s$. Let $x \in I_{\frac{p}{q}}$. $x^q \in \overline{I^p}$ so $v_i(x^q) \geq pv_i(I)$ exactly when $v_i(x) \geq \frac{p}{q}v_i(I)$. Hence the greatest fraction α for which $x \in I_\alpha$ is the least fraction such that $v_i(x) = \alpha v_i(I)$ for some $1 \leq i \leq s$. Therefore the filtration $\{I_\alpha\}$ is a subvalution determined by the Rees valuations of I. □

In particular, the Rees valuations of I are unique by Lemma 3.6, since they are the unique valuations associated to the subvaluation I_α.

From this proof, one can see that the rational powers of an ideal only change in discrete steps. More specifically, let $I_{>\alpha}$ be the ideal of all elements in R which lie in a rational power of I strictly greater than α. Then $I_\alpha/I_{>\alpha} = 0$ for all $\alpha \neq \frac{i}{w}$ for some integer i, where w is the least common multiple of $v_1(I),\dots,v_s(I)$. The consequences of this observation are spelled out in the following theorem.

Theorem 3.10 ([17], 10.5.6)**.** *Let R be a Noetherian ring, let I be an ideal of positive height, and let u be a variable. Let $\{v_1,\dots,v_s\}$ be the Rees valuations of I, and w the least common multiple of $v_1(I),\dots,v_s(I)$. Put $t = u^w$. Let T be the integral closure of the extended Rees algebra $S = R[It,t^{-1}]$ in $R[u,u^{-1}]$.*

(1) T is a $\mathbb{Z}$-graded ring of the form $\oplus_{i\in\mathbb{Z}} J_i u^i$, where J_i are ideals of R and $J_i = R$ for $i \leq 0$.

(2) For $i > 0$, $J_i = I_{\frac{i}{w}}$.

(3) $T/u^{-1}T \simeq \oplus_{\alpha\in\mathbb{Q}_{\geq 0}} I_\alpha/I_{>\alpha}$ is an $\mathbb{N}$-graded algebra with degree zero piece equal to $R/\sqrt{I}$.

(4) $u^{-1}T$ is radical, i.e., $\sqrt{u^{-1}T} = u^{-1}T$.

3.1.1 Noetherian Filtrations and Reductions

As in many aspects of commutative algebra, we impose a finiteness condition on our filtrations, which we then call Noetherian. The most basic finiteness condition is put forth in [2] as a filtration whose associated graded ring $G_f := \oplus_{n\in\mathbb{Z}}(f_n/f_{n+1})$ is a Noetherian ring. We use an alternative condition which appears in the same work as 'essentially powers,' but appears in later work ([14], [28]) as 'Noetherian.' This condition is slightly stronger than the original definition, as was shown in [3] Corollary 3.9.

Definition 3.11. Let R be a ring. A filtration, f, on R is said to be **Noetherian** if there exists an integer d such that $f_n = \sum_{i=1}^{d} f_{n-i} f_i$.

The following theorem provides characterizations of a Noetherian filtration which we find useful.

Theorem 3.12 ([3], 2.2 and 3.6)**.** *Let R be a Noetherian ring. The following conditions are equivalent*

(i) f is a Noetherian filtration

(ii) $\mathscr{R}(f)$ is a Noetherian ring

(iii) $\mathscr{R}(f)$ is finitely generated over R

(iv) There exists a z such that $f_z f_n = f_{n+z}$ for all $n \geq z$

We will frequently use part (iv) of 3.12 to identify a subsequence of $\{f_i\}$ which consists of powers of an ideal: $f_{nz} = f_z^n$ for all n.

A natural definition for the reduction of filtrations is explored by Okon and Ratliff.

Definition 3.13 ([28], 2.1)**.** Consider two Noetherian filtrations f, g on a ring R. We say $g \leq f$ if $g_n \subset f_n$ for all n. We say g is a **reduction** of f if $g \leq f$ and there exists an integer d with $f_n = \sum_{i=1}^{d} g_{n-i} f_i$ for all n.
We say g is a z**-reduction** of f if $g \leq f$ and there $f_{n+z} = g_z f_n$ for n sufficiently large. We say g is a z**-repeating** filtration if $f_{nz+1} = f_{nz+2} = \ldots = f_{(n+1)z} \neq f_{(n+1)z+1}$ for all $n \in \mathbb{N}$.

The following proposition is simply a particular concatenation of results on this subject whose conclusion is useful in Section 3.3.

Proposition 3.14. *If $g \leq f$ is a reduction of Noetherian filtrations, then there exists z such that $g_{nz} = g_z^n$, $f_{nz} = f_z^n$ and $g_z \subset f_z$ is a reduction of ideals.*

Proof. Let $g \leq f$ be Noetherian filtrations. By Theorem 3.12, there exists z_1 such that $f_{nz_1} = f_{z_1}^n$ for all $n \in \mathbb{N}$.

By Theorem 2.9 of [28], there exists z_2 such that $f_{n+z_2} = g_{z_2} f_n$ for all sufficiently large n. Let z be the least common multiple of z_1, z_2.

There exists i, be the least integer such that $f_{iz} \neq g_z f_{(i-1)z}$. Thus, $f_z^n = f_{nz} = g_z^{n-i} f_{iz} = g_z^{n-i} f_z^i$ and g_z may be seen to be a reduction of f_z. $\square$

As we have noted, every Noetherian filtration f has an integer z for which $f_{nz} = f_z^n$. Consider the z-repeating filtration g given by $g_\ell = f_z^{\lceil \frac{\ell}{z} \rceil}$. Here, $\lceil q \rceil = a$ for a rational number q when a is the smallest integer with $a \geq q$. We observe that $g_{n-j} f_j = f_{z \cdot \lceil \frac{n-j}{z} \rceil} f_j \subset f_n$ for any j. Additionally, if $n \geq z$, then consider $n = qz + r$ with $r < z$. Hence $g_{n-z-r} f_{z+r} = f_z^{q-1} f_r = f_n$. Then g is a reduction of f since $\sum_{i=1}^{2z} g_{n-i} f_i = f_n$ for all n.

Further, $f_{p \cdot az} = f_{az}^p$ and $f_{az} \subset f_{bz}$ for any $b < a$, so the az-repeated filtration is a proper reduction of the z-repeated filtration. This shows that every Noetherian filtration will have a proper Noetherian reduction. However, we may still define a minimal z-reduction.

Definition 3.15 ([28])**.** Let g, f be filtrations on a ring R. We say that g is a **minimal z-reduction** of f if for all $n \gg 0$, $g_z f_n = f_{n+z}$ and g has no proper reductions with this property.

Theorem 3.16 ([28], Theorem 2.12)**.** *Let f be a Noetherian filtration in a local ring and let z be a positive integer such that $f_z f_n = f_{z+n}$ for all $n \geq z$. There exists a minimal z-reduction of f.*

The minimal z-reduction in this case is the z-repeating filtration $g_n = J^{\lceil \frac{n}{z} \rceil}$ where J is a minimal reduction of f_z as ideals. Interestingly, a z-repeating reduction appears implicitly in the earlier work of Rees, cf. [31] 9.3.7.

3.1.2 Multiplicity of Noetherian Filtrations

We will review the development of a 'multiplicity' of filtrations which may be seen as a generalization of Hilbert-Samuel multiplicity. These notions will motivate and support our work in section 3.3, which explores a more general multiplicity analogous to the j-multiplicity of Achilles and Manaresi.

The multiplicity of a filtration was first defined in the thesis of Bishop [2] in 1971. Given a filtration f on a ring R, Bishop studies the 'multiplicity' of a filtration f with respect to an R-module M: $e(f,M)$. He defines this as

$$e(f,M) := \lim_{n\to\infty} \frac{d!\lambda(M/f_nM)}{n^d}$$

where the lengths are finite and the limit exists. This 'multiplicity' is not necessarily an integer.

Developments of the theory allow a more concise and comprehensible explanation of these results, so we shall begin with later material. Let $(R,\mathfrak{m})$ be a local ring and let f be a filtration on R with f_1 an $\mathfrak{m}$-primary ideal. Since $f_1^i \subseteq f_i$ for all $i > 1$, each ideal in the filtration is $\mathfrak{m}$-primary. If we assume further that f is Noetherian, then the limit as above exists. Asymptotically, the lengths in question are given by a quasi-polynomial.

Definition 3.17. We say that a function Q defined on the integers is a **quasi-polynomial** of period k if there exist polynomials $\{P_0, P_1, \ldots, P_{k-1}\}$ such that $Q(ak+i) := P_i(ak+i)$ for $0 \leq i \leq k-1$. We say that the degree of Q is $\max_i(\deg(P_i))$.

Q is a **uniform quasi-polynomial** if each polynomial P_i has the same degree and leading coefficient.

If Q is a quasi-polynomial of degree d, then $\lim_{n\to\infty} \frac{1}{n^d} Q(n)$ exists if and only if Q is a uniform quasi-polynomial. In this case, the limit is equal to the common leading coefficient.

Theorem 3.18 ([8], Theorem 2.7)**.** *Let A be an $\mathbb{N}$-graded noetherian ring of finite Krull dimension. Assume that $A = A_0[x_1, \ldots, x_r]$, where each x_i is homogeneous of degree $k_i \geq 1$ and that A_0 is Artinian. Let M be a finitely generated $\mathbb{N}$-graded A-module and let $d = \dim M$. If $k = LCM(k_1, \ldots, k_r)$ then for $n \gg 0$:*

(i) The Hilbert function $H(M,n) := \lambda(M_n)$ is a quasi-polynomial function of period k and degree $d-1$.

(ii) The cumulative Hilbert function $H^\star(M,n) := \sum_{i=0}^n H(M,i)$ is a uniform quasi-polynomial of degree d and period k.

Corollary 3.19. *If $(R, \mathfrak{m})$ is a local ring of Krull dimension d, f is a Noetherian filtration, and f_1 is an $\mathfrak{m}$-primary ideal, then $\lambda(f_n/f_{n+1})$ is given by a quasi-polynomial of degree $d-1$ and $\lambda(R/f_n)$ is given by a uniform quasi-polynomial of degree d for $n >> 0$.*

Proof. Since f is Noetherian, the associated graded ring $G_f := \oplus f_i/f_{i+1}$ is a finitely generated algebra over R/f_1. Moreover, f_1 is $\mathfrak{m}$-primary, so G_0 is Artinian. The result now follows from Theorem 3.18. $\square$

Definition 3.20. Let $(R, \mathfrak{m})$ be a local ring and let f be a Noetherian filtration with f_1 an $\mathfrak{m}$-primary ideal. We may define the multiplicity of f with respect to an R-module M by

$$e(f, M) := \lim_{n\to\infty} \frac{d!}{n^d} \lambda(M/f_n M)$$

Example 3.21. Let $R = k[x, y]$ and consider the 3-repeating filtration based on powers of $I = (x^2, y)$. That is, let f be the filtration on R with $f_{3i+1} = f_{3i+2} = f_{3i+3} = I^{i+1}$.

The sequence $\lambda(f_i/f_{i+1})$ for $i \in \mathbb{N}$ is given by $\lambda(R/I), 0, 0, \lambda(I/I^2), 0, 0, \lambda(I^2/I^3), \ldots$. Since $\lambda(I^i/I^{i+1}) = 2i+2$ we see that $\lambda(f_i/f_{i+1})$ is a quasi-polynomial of period 3. The three polynomials defining $\lambda(f_i/f_{i+1})$ are $\{p_0(i) = 2\frac{i}{3} + 2, p_1(i) = 0, p_2(i) = 0\}$.

Consider $\lambda(R/f_n) = \sum_{i=0}^{n-1} \lambda\left(\frac{f_i}{f_{i+1}}\right)$. This quasi-polynomial of period 3 is given by

$$\{P_0 = \sum_{i=0}^{\frac{n}{3}} (2i+2), P_1 = \sum_{i=0}^{\frac{n-1}{3}} 2i+2, P_2 = \sum_{i=0}^{\frac{n-2}{3}} 2i+2\}$$

By applying the identity $\sum_{i=1}^{a} i = \frac{i(i+1)}{2}$ in each case, we simplify to

$$\begin{aligned}
P_0 &= \frac{1}{9}n^2 + n \\
P_1 &= \frac{1}{9}(n-1)^2 + (n-1) = \frac{1}{9}n^2 + \frac{7}{9}n - \frac{8}{9} \\
P_2 &= \frac{1}{9}(n-2)^2 + (n-2) = \frac{1}{9}n^2 + \frac{5}{9}n - \frac{14}{9}
\end{aligned}$$

This is a uniform quasi-polynomial with leading coefficient $\frac{1}{9}$, so $e(f) = \frac{2}{9}$.

The following formula for the multiplicity of Noetherian $\mathfrak{m}$-primary filtrations appears in the earliest work on the subject. Our proof emphasizes the modern notation.

Theorem 3.22 ([2], 2.22)**.** *Let f be a Noetherian filtration on a local ring $(R,\mathfrak{m})$. Further suppose that f_1 is $\mathfrak{m}$-primary so that $\lambda(R/f_i) < \infty$ for all i. For any z such that $f_z f_n = f_{n+z}$ for all $n \geq z$, $e(f) = \frac{e(f_z)}{z^d}$, where $e(f_z)$ is the Hilbert-Samuel multiplicity of the ideal f_z.*

Proof. Form the associated graded ring of the filtration: $G(f) = \mathscr{R}(f)/t^{-1}\mathscr{R}(f)$. $G(f)$ is finitely generated as an algebra over the Artinian ring R/f_1, so the length of its graded pieces f_i/f_{i+1} is eventually a quasi-polynomial of degree $d-1$. Hence $\lambda(R/f_n)$ is eventually given by a uniform quasi-polynomial of degree d. Now consider blocks of the grading of size z:

$$\sum_{i=ka+1}^{(k+1)z} \lambda(f_{i-1}/f_i) = \lambda\left(\frac{f_{kz}}{f_{(k+1)z}}\right) = \lambda\left(\frac{f_z^k}{f_z^{k+1}}\right)$$

This gives

$$\lim_{n\to\infty} \frac{d!\lambda(R/f_n)}{n^d} = \lim_{n\to\infty} \frac{d!\lambda(R/f_{zn})}{(zn)^d} = z^{-d} \cdot \lim_{n\to\infty} \frac{d!\lambda(R/f_z^n)}{(n)^d} = z^{-d} \cdot e(f_z)$$

□

In the Theorem 3.22, it is worth noting that the z for which $f_{nz} = f_z^n$ is not unique. Any choice of such a z will result in the same multiplicity, as may be inferred from a quick calculation. For example, if $f_{nz} = f_z^n$, then $f_{n\cdot 2z} = f_{2z}^n$. But $e(f_{2z}) = e(f_z^2) = 2^d e(f_z)$. Hence $e(f_{2z})/(2z)^d = e(f_z)/z^d$. We extend this to a more general observation on the multiplicity of a subsequence of a filtration in Proposition 3.47.

There again occurs a natural associativity formula for this multiplicity; there is no substantial change in the proof from the ideal case.

Proposition 3.23 ([31], 8.2 (iv))**.** *Let R be a local ring, M a finitely generated R-module and f a*

Noetherian filtration of $\mathfrak{m}$*-primary ideals. Let* $\Lambda = \{P \subset R \text{ prime} | \dim(R/P) = \dim(R)\}$.

$$e(f,M) = \sum_{P \in \Lambda} e(f,R/P)\lambda(M_P)$$

In a further analogue of the ideal case, we wish to associate to a filtration certain rank one discrete valuation rings. The remaining material of this section may be found in the work of Rees [31], though the presentation is different so that specific citations are not possible for every result.

Definition 3.24 ([31])**.** Let R be a domain. We may form a valuation ring by taking the integral closure of $\mathscr{R}(f)$ in its field of fractions and localizing at a prime minimal over (t^{-1}). We say V is a **Rees valuation ring of a filtration** f on R when V is obtained by the intersection of such a ring with the fraction field of R.

When R is not a domain, we may identify the Rees valuations of f by applying this construction to the image of f on R/P for each minimal prime P of R.

One may see that if f is the filtration of powers of f_1, then V is a Rees valuation of f_1 and this definition is a generalization of the Rees valuations of an ideal. We shall see that the multiplicity of the filtration may be expressed in terms of these valuation rings in Theorem 3.49.

Proposition 3.25 ([31])**.** *Let f be a Noetherian filtration on a domain R. Let z be a positive integer with $f_{nz} = f_z^n = I^n$. The set of Rees valuations of f is the set of Rees valuations of $I = f_z$.*

Proof. We may find the Rees valuations of I by examining $R[t^{-1}, It]$. Each such valuation is formed by localizing the integral closure of $R[t^{-1}, It]$ at a prime minimal over (t^{-1}). Now consider $R[t^{-z}, It^z]$. Since no power of t which is not a multiple of z lies in the quotient field of this domain, the ring isomorphism, $\psi : R[t^{-1}, It] \to R[t^{-z}, It^z]$ with $\psi(t^{-1}) = t^{-z}$ and $\psi(x) = x$ for all $x \in R$ extends to an isomorphism, ψ', on the respective integral closures. Therefore, the prime ideals have a one-to-one correspondence. Let P be a prime of $\overline{R[t^{-1}, It]}$ and let $P' = \psi'(P)$. Then the valuation rings $V = \overline{R[t^{-1}, It]}_P$ and $V' = \overline{R[t^{-z}, It^z]}_{P'}$ are isomorphic. When we omit t by intersecting with the fraction field K of R, we find $\{V \cap K\} = \{V' \cap K\}$. In particular, this holds for the primes P minimal over t^{-1} so the Rees valuations of I may be calculated from either ring.

Next, note that $S_1 := R[t^{-z}, It^z] \subset \mathscr{R}(f) := S_2$ is an integral extension of rings. $\overline{S_1}$ is an integrally closed domain and $\overline{S_2}$ is an integral extension of $\overline{S_1}$. Therefore, we may apply the Incomparability and Going Down theorems ([17], 2.2.3 and 2.2.7) to prove that $Q \subset \overline{S_2}$ is a minimal prime over $(t^{-1}$ if and only if $P = Q \cap \overline{S_1}$ is a minimal prime over (t^{-z}).

Suppose $Q \subset \overline{S_2}$ is a prime with $P = Q \cap \overline{S_1}$ a minimal prime over (t^{-z}). If Q' is a prime ideal with $(t^{-1}) \subset Q' \subset Q$, then $P' = Q' \cap \overline{S_1}$ is a prime ideal of $\overline{S_1}$ with $(t^{-z}) \subset P' \subset P$ so that $P' = P$. By incomparability, $Q' \cap \overline{S_1} = Q \cap \overline{S_1}$ and $Q' \subset Q$ implies $Q' = Q$. Hence Q is minimal over (t^{-1}).

Suppose there exists a prime $P' \neq P$ such that $(t^{-z}) \subset P' \subset P = Q \cap \overline{S_1}$. By going down, there exists a chain of primes $Q' \subset Q$ with $Q' \cap \overline{S_1} = P'$. Since $(t^{-z}) \in P'$, $(t^{-1}) \in Q'$ and Q is not a minimal prime over (t^{-1}).

Letting K' denote the fraction field of S_1, we have $(\overline{S_2})_Q \cap K' = (\overline{S_1})_P$. Hence the Rees valuations of the filtration, formed by $(\overline{S_2})_Q \cap K$ for Q minimal over (t^{-1}), are isomorphic to the valuations $(\overline{S_1})_P \cap K$ which we have already shown are the Rees valuations of I. $\square$

The following theorem generalizes Rees' association between the multiplicity and the valuations of an ideal, recorded in 1.27, to the case of $\mathfrak{m}$-primary Noetherian filtrations.

Theorem 3.26 ([31], 9.31)**.** *Let $(R, \mathfrak{m}, k)$ be a local domain. Let f be a Noetherian filtration of $\mathfrak{m}$-primary ideals. There exist positive rational numbers $d(f, v_i)$ such that*

$$e(f, R) = \sum_{i=1}^{s} d(f, v_i) v_i(x)$$

for $v_1, \ldots, v_s$ Rees valuations of f.

3.2 Are Rees Valuations Noetherian?

Noetherian filtrations have come under considerable study. In this section we seek to add to the number of concrete classes of filtrations which are known to have this property. In particular, there are two natural questions which follow from the observations of Lemma 3.6 and Proposition

3.9. If f is a Noetherian subvaluation, are the valuations uniquely associated to f also Noetherian filtrations? And, what is in most cases a specialization of this, are Rees valuations of an ideal in a Noetherian ring Noetherian filtrations?

In this section we investigate the second question: Let R be a Noetherian local ring and let $I \subset R$ be an ideal. Let V be a Rees valuation ring of I with maximal ideal $\mathfrak{m}_V$. The corresponding valuation on R is a filtration, f, with ideals $f_n = \mathfrak{m}_V^n \cap R$. When can we say that f is a Noetherian filtration?

When I is a monomial ideal in a polynomial ring R, it is known that the ideals $\mathfrak{m}_V \cap R$ are also monomial. We offer a variation on the proof of this fact in Theorem 3.30 and note that this implies that the filtration is Noetherian. In Theorem 3.32, we identify a strong set of conditions on an ideal which allow us to directly calculate the ideals $\mathfrak{m}_V^n \cap R$ and show that the Rees valuation is a Noetherian filtration. Finally we explore a technique of constructing Noetherian rings which in some cases are the Rees rings of Rees valuations, culminating in Theorem 3.38.

Before this, we make a general observation concerning the Rees valuations of a valuation. Recall that we define the Rees valution of a filtration in 3.24.

Proposition 3.27. *Let v be a valuation on a domain R. As a filtration, v has one Rees valuation, which is v.*

Proof. Let $f_i = \{x \in R | v(x) \geq i\}$ and consider the Rees ring: $\mathscr{R}(v) = R[t^{-1}, f_1 t, f_2 t^2, \ldots]$. Since v is a valuation, $t^{-1}\mathscr{R}(v)$ is a prime ideal. Localizing at t^{-1} inverts elements of the form $xt^{v(x)}$ for $x \in R$. We consider $\mathscr{V} = \mathscr{R}(v)_{(t^{-1})} \cap K$ where K is the fraction field of R.

Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{V}$, $\mathfrak{m}^s = (t^{-s}\mathscr{R}(v) \cap K)$. Since $x \in R$ has $x \in \mathfrak{m}^s$ if and only if $v(x) \geq s$, $\mathscr{V}$ is the valuation ring of v. $\square$

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring. For each Rees valuation v of $I = (r_1, \ldots, r_n) \subset R$, there exists a choice of generator r_i and prime $P \in Spec(\overline{R[\frac{I}{r_i}]})$ such that $\overline{R[\frac{I}{r_i}]}_P = V$, the valuation ring of v. We will prove that if I is a monomial ideal, then $\mathfrak{m}_V^n \cap R$ is a monomial ideal for each n. First, an example to demonstrate that the assumption of monomial on I is necessary.

Example 3.28. Let $R = k[x,y]$ and let $I = (x^2, x+y, y^2)$. We calculate $R[I/x^2] = k[x,y,\frac{x+y}{x^2},\frac{y^2}{x^2}]$. One may see that $\frac{y}{x}$ lies in the integral closure of $R[I/x^2]$; simplifying $R[I/x^2, y/x]$ we arrive at $k[x,\frac{x+y}{x^2},\frac{y}{x}]$. Since $\frac{y}{x} = x\frac{x+y}{x^2} - 1$, $k[x,\frac{x+y}{x^2},\frac{y}{x}] = k[x,\frac{x+y}{x^2}]$. Furthermore, since x, y are independent variables, these two generators have no relations.

We have $\overline{R[I/x^2]} = k[x,\frac{x+y}{x^2}]$ is a regular ring with (x) a height one prime. We localize at (x) to find the Rees valuation ring V. The valuation of x is 1, and since $y = x(x\frac{x+y}{x^2} - 1)$, $v(y) = 1$ as well. However, $x+y = x + (x^2\frac{x+y}{x^2} - x) = x^2\frac{x+y}{x^2}$ has value 2. Since $x, y \notin \mathfrak{m}_V^2 \cap R$ and $x+y \in \mathfrak{m}_V^2 \cap R$, this ideal is not homogeneous.

Put another way, $\mathscr{R}(v) = R[t^{-1}, xt, (x+y)t^2]$. Hence $v_2 = t^{-2} \cap R \cong (x^2, x+y)$, which is not a monomial ideal.

The following statement is known, though we provide our own proof using a 'homogeneous localization'.

Proposition 3.29 ([17], 10.3.4). *Let $R = k[x_1, \ldots, x_d]$ and let $I = (m_1, \ldots, m_t)$ be a monomial ideal. Let v be a Rees valuation of I. The valuation ideals $f_n = \{r \in R | v(r) \geq n\}$ are monomial.*

Proof. Define $S = R[\frac{I}{m_1}]$, take the integral closure $\overline{S}$ of S. Then for any prime P minimal over $m_1\overline{S}$, $\overline{S}_P = V$ is one of the Rees valuations of I. We establish a grading on S and use this to show that $f_n = P^n V \cap R$ is a monomial ideal.

Since m_1 is a monomial, S is contained in $k[x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}]$ and inherits its natural $\mathbb{Z}^d$ grading.

This grade descends to R, in which an ideal is homogeneous if and only if it is monomial. Since S is a domain, the grading extends to $\overline{S}$ by [17] Proposition 2.3.5.

The principal ideal $m_1\overline{S}$ is $\mathbb{Z}^d$-graded, so the minimal primes containing it will be homogeneous. Let $P = (\pi, r_1, \ldots, r_s) \subset \overline{S}$ be one of these primes, where $\pi, r_1, \ldots, r_s$ are a set of homogeneous generators. Let $\mathscr{U}$ be the set of homogeneous elements of $\overline{S}$ which are not in P, and take the localization $\overline{S}_{\mathscr{U}}$. Since we invert only elements which are homogeneous, we preserve the $\mathbb{Z}^d$-grading of $\overline{S}$.

We claim that $P\overline{S}_{\mathscr{U}}$ is principal. For the discrete valuation ring $V := \overline{S}_P$, PV is principal; we may assume it is generated by π. For each generator, r_i of P, there exists $\frac{s_i}{q_i} \in V$ such that $r_i = \pi \cdot \frac{s_i}{q_i}$ for some $q_i \notin P$. Clearing the denominator gives a relation in $\overline{S}$: $r_i \cdot q_i = \pi \cdot s_i$. Because $\pi\overline{S}$ is a homogeneous ideal, each homogeneous component of q_i is in $(\pi :_{\overline{S}} r_i)$. Since at least one homogeneous component, q_i', is not in P, we have $q_i' \notin P$ with $q_i' \cdot r_i \in \pi\overline{S}$. This implies $r_i \in \pi\overline{S}_{\mathscr{U}}$. Therefore, $P\overline{S}_{\mathscr{U}} = \pi\overline{S}_{\mathscr{U}}$ is principal.

Since every homogeneous element outside P has been inverted $P\overline{S}_{\mathscr{U}}$ contains every homogeneous prime of $\overline{S}_{\mathscr{U}}$. In particular, P is the only homogeneous height 1 prime of $\overline{S}_{\mathscr{U}}$. Being normal, $\overline{S}_{\mathscr{U}}$ satisfies S_2 and thus $\pi^n\overline{S}_{\mathscr{U}}$ is P-primary. We may complete the localization at P, giving $(\pi^n\overline{S}_{\mathscr{U}})_P = \pi^n\overline{S}_P = \pi^n V$. Since $\pi^n\overline{S}_{\mathscr{U}}$ is P-primary, $\pi^n\overline{S}_{\mathscr{U}} = \pi^n V \cap \overline{S}_{\mathscr{U}}$. Now $f_n = \pi^n V \cap R = \pi^n\overline{S}_{\mathscr{U}} \cap R$. Since $\overline{S}_{\mathscr{U}}$ is graded and $\pi^n\overline{S}_{\mathscr{U}}$ is homogeneous, f_n is homogeneous. In R, a homogeneous ideal in this $\mathbb{Z}^d$-grading is monomial. $\square$

Theorem 3.30. *Let $I \subset k[x_1, \ldots, x_d]$ be a monomial ideal and let v be a Rees valuation of I. Then v is a Noetherian filtration.*

Proof. We will show that $\mathscr{R}(v) = R[t^{-1}, x_1 t^{v(x_1)}, \ldots, x_d t^{v(x_d)}]$. By Proposition 3.29, $\mathscr{R}(v)$ may be generated in each degree by a finite number of monomials. Since v is a valuation, the valuation of any monomial is given by $v(\prod_{i=1}^d x_i^{p_i}) = \sum_{i=1}^n p_i \cdot v(x_i)$. But this shows that every term of $\mathscr{R}(v)$ may be generated over $R[t^{-1}]$ by $x_1 t^{v(x_1)}, \ldots, x_d t^{v(x_d)}$. Therefore, v satisfies condition (iii) of Theorem 3.12. $\square$

Given a more general ideal, the filtration ideals will no longer be monomial, but we may still investigate whether the resulting filtration is Noetherian. We return to Example 3.28 and directly calculate the sequence of ideals associated to a Rees valuation.

Example 3.31. Let $R = k[x,y]$, $I = (x+y, x^2)$ and consider the Rees valuation of I found by adjoining $\frac{x+y}{x^2}$ to R. We may observe that the Rees valuation will be a Noetherian filtration since I may be made monomial by a change of variable. Here we demonstrate the same fact by calculating the filtration ideals.

Consider $R[\frac{x+y}{x^2}]$. Since $y = x^2\left(\frac{x+y}{x^2}\right) - x$, $R[\frac{x+y}{x^2}] = k[x, \frac{x+y}{x^2}]$ which is isomorphic to a polynomial ring over k in two variables. Hence $R[\frac{x+y}{x^2}]$ is regular and normal.

The localization of $R[\frac{x+y}{x^2}]$ at a height 1 prime containing $x^2R[z]$ is the Rees valuation ring $(V, \mathfrak{m})$. This prime, $xR[\frac{x+y}{x^2}]$, is a principal prime, so its powers are all $xR[\frac{x+y}{x^2}]$-primary. In particular, $\mathfrak{m}^nV \cap R[z]$, which defines the n^{th} symbolic power of $xR[\frac{x+y}{x^2}]$, is simply $x^nR[\frac{x+y}{x^2}]$.

We seek the filtration of ideals generated by this valuation, namely

$$f_n = \mathfrak{m}^nV \cap R = \mathfrak{m}^nV \cap R[\frac{x+y}{x^2}] \cap R = x^nR[\frac{x+y}{x^2}] \cap R$$

We characterize $R[\frac{x+y}{x^2}]$ as $\cup_{d=0}^{\infty}\frac{I^d}{x^{2d}}$. This characterization shows $a \in x^nR[\frac{x+y}{x^2}] \cap R$ if and only if $x^nI^d = ax^{2d}$ for some d. Thus, $x^nR[\frac{x+y}{x^2}] \cap R = \cup_{d=0}^{\infty}(x^nI^d : x^{2d})$. Fixing n, the ideals in the union are an ascending chain of ideals in R. Thus, the union is their stable value. Moreover, for $d > n/2$, this simplifies to $(I^d : x^{2d-n})$ since R is a domain.

Because I is generated by a regular sequence, we also have

$$\begin{aligned}(I^d : x^{2d-n}) =& ((x^{2d}, x^{2d-2}(x+y), \ldots, (x+y)^d) : x^{2d-n}) \\ =& (x^{2d} : x^{2d-n}) + (x^{2d-2}(x+y) : x^{2d-n}) + \ldots + ((x+y)^d : x^{2d-n})\end{aligned}$$

For n odd, $(I^d : x^{2d-n}) = (x^n) + (x^{n-2}(x+y)) + \ldots + (x(x+y)^{\frac{n-1}{2}}) + (x+y)^{\frac{n+1}{2}}$. Grouping the terms with a factor of x, we may express this as $xI^{\frac{n-1}{2}} + (x+y)^{\frac{n+1}{2}} = xI^{\frac{n-1}{2}} + I^{\frac{n+1}{2}}$. Similarly, for n even, $(I^d : x^{2d-n}) = I^{\frac{n}{2}}$. This gives $\mathfrak{m}^2V \cap R = I$ and $I(\mathfrak{m}^nV \cap R) = \mathfrak{m}^{n+2}V \cap R$ for all n. Therefore, this filtration is Noetherian by 3.12 (iv).

We now give a description of the general Rees valuation which may be determined to be Noetherian by calculating the valuation ideals in this manner.

Theorem 3.32. *Let R be a Noetherian normal domain and let $I = (a_1, \ldots, a_n, b)$. Suppose that $S := R[\frac{I}{b}]$ is integrally closed and there exists $\beta \in R$ with $\beta^p = b$ and βS a prime ideal. Suppose further that for $0 < r \leq p$ and for all $d \in \mathbb{N}$, $(I^d : \beta^r) = \beta^{p-r}I^{d-1} + I^d$. Then the Rees valuation of*

I obtained by localizing S at βS is a Noetherian filtration.

Proof. Let V be the valuation ring obtained by localizing S at βS. We directly calculate $\mathfrak{m}_V^n \cap R$.

The maximal ideal of V is generated by the image of β. Hence $\mathfrak{m}_V^n \cap S = \beta^n S_{(\beta)} \cap S = (\beta S)^{(n)}$, the n^{th} symbolic power of the prime βS. Since βS is a principal prime, its symbolic powers are equal to its powers. This allows us to rewrite the original ideals as $\mathfrak{m}_V^n \cap R = \mathfrak{m}_V^n \cap S \cap R = \beta^n S \cap R$.

Consider $S = \cup_{d=0}^{\infty} \frac{I^d}{\beta^{pd}}$. We observe from this that $a \in \beta^n S \cap R$ if and only if $a\beta^{pd} \in \beta^n I^d$ for some d. We may characterize the valuation ideals by $(\beta)^n S \cap R = \cup_{d=0}^{\infty} (\beta^n I^d : \beta^{pd})$. This is the union of an increasing sequence of ideals in d, so the union is equal to its stable value. For $pd > n$, we may write $\mathfrak{m}^n V \cap R = (I^d : \beta^{pd-n})$.

To show that $f_n := (I^d : \beta^{pd-n})$ is a Noetherian filtration, we claim that $f_p f_n = f_{n+p}$ for all $n \geq p$ as in (iv) of Theorem 3.12.

Note that by hypothesis with $r = p$, $(I^d : \beta^p) = I^{d-1}$ for any d.

Let $n = pq - r$ for integers q, r with $0 < r \leq p$. Rewrite $(I^d : \beta^{pd-n})$ as $(I^d : \beta^{p(d-q)+r})$. If $\alpha \in (I^d : \beta^{p(d-q)+r})$, then

$$\alpha \cdot \beta^{p(d-q-1)+r} \in (I^d : \beta^p) = I^{d-1}$$

Hence $\alpha \in (I^{d-1} : \beta^{p(d-q-1)+r})$. Applying this reduction $d-q$ times leaves $(I^q : \beta^r)$. That is, for $n = pq - r$, $f_n = (I^q : \beta^r)$.

Let $n = p$. Then $p = 2p - p$ has $q = 2$, $r = p$ and $f_p = (I^2 : \beta^p)$. Our hypothesis gives us $f_p = (I^2 : \beta^p) = \beta^0 I + I^2 = I$.

Let $n = pq - r > p$ so that $q \geq 2$. By our characterization above, $f_{n+p} = (I^q : \beta^r)$. Our hypothesis gives $(I^q : \beta^r) = \beta^{p-r} I^q + I^{q+1}$. Thus, $I(\beta^{p-r} I^{q-1} + I^q) = f_p f_n$ by factoring I out of both terms. Since $f_{n+p} = f_n f_p$ for all $n \geq p$, this filtration is Noetherian. □

Example 3.33. Let $R = \mathbb{Q}[x,y,z]$ and consider $I = (x^2, y^2 + z^2)$. Letting $b = x^2$, we see $(I^d : x) = xI^{d-1} + (y^2 + z^2)^d$ and $(I^d : x^2) = I^{d-1}$. Hence the associated filtration is Noetherian.

However, if $b = y^2 + z^2$, then bS is not prime so we may not conclude that the valuation associated to $R[\frac{x^2}{y^2+z^2}]$ is Noetherian.

As one can see above, the method of directly calculating the ideals is only productive in the context of strong hypotheses. We explore another method which is based on constructing graded rings which 'ought' to be the Rees rings of Rees valuations.

In Proposition 3.9, we established that the rational powers formed a subvaluation. It would seem natural that the valuations determined by a Noetherian subvaluation would all be Noetherian, but this is not known. In the following work, our goal is to construct a collection of Noetherian filtrations from the subvaluation of rational powers. These filtrations are the Rees valuations of I if and only if they are valuations. The following is a helpful condition for recognizing when a Noetherian filtration, defined in terms of its Rees algebra, is a valuation.

Lemma 3.34. *Let R be a local ring and P a prime ideal which may be generated by a regular sequence: $P = (a_1, \dots, a_n)$. Then $\mathscr{R} = R[t^{-1}, a_1 t^{p_1}, \dots, a_n t^{p_n}]$ is the Rees ring of a discrete valuation over R.*

Proof. We may define a filtration f from $\mathscr{R}$ by $f(x) = n$ if and only if $xt^n \in \mathscr{R}$ and $xt^{n+1} \notin \mathscr{R}$. It remains to show that $f(xy) = f(x) + f(y)$ for all $x, y \in R$. This is equivalent to saying that $t^{-1}\mathscr{R}$ is a prime ideal of $\mathscr{R}$.

Let $S = R[T, U_1, \dots, U_r]$ be the R-algebra with $r+1$ independent variables and consider the R-linear homomorphism $\psi : S \to \mathscr{R}$ given by $\psi(T) = t^{-1}$ and $\psi(U_i) = a_i t^{p_i}$. Let K be the kernel of ψ.

Define $C := (T^{p_1}U_1 - a_1, \dots, T^{p_r}U_r - a_r)$ and note that C is a submodule of K. Consider a monomial $\mu = \prod_{i=1}^r U_i^{w_i}$. Rewrite $T^{\sum_i p_i w_i}\mu$ by replacing each $T^{p_i}U_i$ with $(T^{p_i}U_i - a_i + a_i)$. The monomial may then be expressed as $H + \prod_{i=1}^r a_i^{w_i}$ where $H \in C$. Hence, any homogeneous form $F(U_1, \dots, U_t)$ has $T^n F(U_1, \dots, U_r) = H + F(a_1, \dots, a_r)$ with $H \in C$. If we further assume that $F \in K$, then we see that $F(a_1, \dots, a_r) = 0$. Thus, $K = (C : T^\infty)$.

This argument that $K = (C : T^\infty)$ does not require that $a_1, \dots, a_r$ be a regular sequence. We claim that when the sequence is regular, $C = K$.

Let $P \subset S$ be a prime ideal containing C. If $T \in P$, then $a_1, \dots, a_r, T \in P$ and S_P has depth at least $r+1$. If $T \notin P$, then $U_i - \frac{a_i}{T^{p_i}} \in P$ so $\operatorname{depth}(S_P) \geq r$.

Since C is generated by r elements, this also shows that C must be generated by a regular sequence. For any $P \in \operatorname{Ass}(S/C)$, we have $\operatorname{depth}(S_P) = r$. Hence T is not contained in any associated primes of S/C, so $K = (C : T^\infty) = C$.

Now let $I \subset R$ be the ideal generated by $(a_1, \ldots, a_r)$. Since $K = C$, $K + (T) = I + (T)$ as ideals in S. Therefore, $\mathscr{R}/t^{-1}\mathscr{R} \cong S/(K + (T))S \cong R/I[U_1, \ldots, U_r]$. This last ring is r variables adjoined to a domain R/I, so it is a domain. In particular, $t^{-1}\mathscr{R}$ is a prime ideal of $\mathscr{R}$. $\square$

We now assume that R is a UFD and construct the Rees ring of a new filtration based on the irreducible factors of generators of I.

It is worth noting that our established notations for filtrations and for valuations leads to a possibly confusing juxtaposition. Given two filtrations, we have said $g \leq f$ if $g_i \subseteq f_i$ for all i. This is equivalent to $f(x) \leq g(x)$ for any $x \in R$. We continue with this incongruity in mind.

Construction 3.35. Let $(R, \mathfrak{m})$ be a local UFD and let f be a Noetherian filtration on R. Let $\mathscr{R}(f) = R[t^{-1}, q_1 t^{p_1}, \ldots, q_n t^{p_n}]$ and let $a_1, \ldots, a_m$ be the set of all irreducible factors of the elements $q_1, \ldots, q_n$. We write $q_i = \prod_{j=1}^m a_j^{c_{ij}}$. We will construct filtrations $h_1, \ldots, h_s$ with constants $w_1, \ldots, w_s$ such that $h_\ell \leq w_\ell f$ and $h_\ell(q_i) = \sum_{j=1}^m c_{ij} h_\ell(a_j)$ for all ℓ, i.

Any such filtration h_ℓ satisfies $n+m$ linear inequalities on the m unknowns $h_\ell(a_1), \ldots, h_\ell(a_m)$ given by $h_\ell(q_i) = \prod_{j=1}^m c_{ij} h_\ell(a_j) \geq w f(q_i)$ and $h_\ell(a_j) \geq w f(a_j)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

From these $n+m$ inequalities, we form an $(n+m) \times (m+1)$ matrix. For $1 \leq i \leq m$, column i will correspond to coefficients of $h_\ell(a_i)$ and column $m+1$ will be given by $f(q_i)$ or $f(a_i)$ as appropriate. For example, the row that corresponds to $h_\ell(q_1) = \prod_{j=1}^m c_{1j} h_\ell(a_j) \geq w f(q_1)$ will be given by $\left[\begin{smallmatrix} c_{11} & c_{12} & \cdots & c_{1m} & f(q_1) \end{smallmatrix}\right]$.

Fix one of the $\binom{n+m}{m}$ choices of m rows to place at the top of this matrix and name the resulting matrix M. Consider the $m \times m$ submatrix A in the top left corner; that is, with $A_{ij} = M_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq m$. Apply row operations (over $\mathbb{Q}$) to the whole matrix M which place A in reduced row echelon form. We will assume that A is invertible so that we have the $m \times m$ identity matrix. Note that although all entries were integers at the start, column $m+1$ may now contain rational entries. After this row reduction, form M' by adding multiples of rows 1 through m to the remaining rows to obtain an $n \times m$ zero submatrix; the entries of column $m+1$ may not be zero. Assume that $M'_{i(m+1)} \leq 0$ for $m < i \leq n+m$.

Now suppose we have a rational function α on R with $\alpha(a_i) = M'_{i(m+1)}$ for $1 \leq i \leq m$ and $\alpha(q_i) = \sum_{j=1}^{m} c_{ij}\alpha(a_j)$. This function satisfies the inequalities associated to the first m rows of M with equality and satisfies the remaining n inequalities since they are equivalent to $0 \geq M'_{i(m+1)}$ for $m < i \leq n+m$. Let w be the least common multiple of the denominators of $M'_{i(m+1)}$ for $1 \leq i \leq m$. Then let g be defined as the filtration whose Rees ring is

$$\begin{aligned}\mathscr{R}(g) =& R[t^{-1}, q_1 t^{wp_1}, \ldots, q_n t^{wp_n}, a_1 t^{wM'_{1(m+1)}}, \ldots, a_m t^{wM'_{m(m+1)}}] \\ =& R[t^{-1}, a_1 t^{wM'_{1(m+1)}}, \ldots, a_m t^{wM'_{m(m+1)}}]\end{aligned}$$

Thus $g \leq wf$ and $g(q_i) = \sum_{j=1}^{m} c_{ij} g(a_j)$ by construction.

For each choice of m initial rows, we may be able to construct such a filtration. If a given choice of rows results in a submatrix A which is not invertible or if $M'_{i(m+1)} > 0$ for some $i > m$, then we omit this choice of initial rows from consideration. After computing such matrices for every choice of initial rows, we will have a finite set of filtrations $h_1, \ldots, h_s$ with associated integers $w_1, \ldots, w_s$ for which $h_\ell \leq w_\ell f$ and $h_\ell(q_i) = \sum_{j=1}^{m} c_{ij} h(a_j)$ for all ℓ, i.

Example 3.36. Let $R = k[x,y]$ and consider $\mathscr{R}(f) = R[t^{-1}, xyt]$. In this case, $n = 1$ and $q = xy$. It has two irreducible factors, $a_1 = x$, $a_2 = y$. These take the values $f(xy) = 1, f(x) = 0$ and $f(y) = 0$.

We form M as above and find

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There are 3 choices of which rows may be the top two, all of which give an A that reduces to I_2.

$$M_1' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad M_2' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad M_3' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first two matrices identify two filtrations h_1, h_2 whose Rees rings are $R[t^{-1}, yt]$ and $R[t^{-1}, xt]$, respectively. Note that h_1, h_2 are the Rees valuations of the ideal (xy) as we showed in Example 3.4. The third matrix suggests the filtration h_3 with $h_3(x) = h_3(y) = h_3(xy) = 0$, but the positive 1 at $(M_3')_{33}$ indicates that this choice does not satisfy our initial inequalities.

Lemma 3.37. *Let R be a UFD and let f be a Noetherian filtration on R. Then Construction 3.35 produces at least one filtration.*

Proof. Denote $\mathscr{R}(f) = R[t^{-1}, q_1 t^{p_1}, \ldots, q_n t^{p_n}]$ for some $q_i \in R$ and $p_i \in \mathbb{N}$. Let $\{a_1, \ldots, a_m\}$ be the union of irreducible factors of each q_i; $q_i = \prod_{j=1}^{m} a_j^{c_{ij}}$.

We need only show that we may produce a matrix M whose top left $m \times m$ submatrix is invertible and for which the reduced matrix M' has $M'_{i(m+1)} \leq 0$.

We may interpret the problem geometrically by associating a function g having $g(a_j) = r_j$ to $(r_1, \ldots, r_m) \in \mathbb{Q}^m$. In order to have $g(a_j) \geq f(a_j)$ and $g(q_i) = \sum_{j=1}^{n} c_{ij} g(a_j) \geq f(q_i)$, we choose a point in the intersection of $n+m$ half-spaces. Since every $c_{ij} \geq 0$, choosing $g(a_j) = \max_i(f(q_i))$ for all $1 \leq j \leq m$ gives one such point. Hence we have a nonempty polytope of solutions in $\mathbb{Q}^m$. In order to have a matrix of the required form, we simply need the boundary of this polytope to contain a vertex, which will be the intersection of at least m hyperplanes. The $(n+m) \times m$ matrix of coefficients has full rank, so such a point exists. □

Recall in Theorem 3.10 we found that the set of Rees valuations, $\{v_1, \ldots, v_s\}$, of an ideal I is the unique set of valuations defining the subvaluation, f, of rational powers of I. For each valuation v_i, there is a corresponding set $S_i \in R$ such that $v_i(x) = f(x)$ if and only if $x \in S_i$. The sets $S_1, \ldots, S_s$ are exactly the maximal f-compatible sets of R. We will use these notions to prove our next theorem.

Theorem 3.38. *Let I be an ideal in an unramified Noetherian unique factorization domain R. Let f be the filtration of rational powers of I. We may construct a finite set of Noetherian filtrations $h_1,\ldots,h_s$ such that if h_j is a valuation then h_j is a Rees valuation of I.*

Proof. Since R is unramified, the Rees algebra of rational powers of I is a finitely generated algebra over R by [17] Theorem 9.2.2. Let w be the least common multiple of $\{v_1(I),\ldots,v_s(I)\}$ where $v_1,\ldots,v_s$ are the Rees valuations of I. The rational powers of I are given by $\overline{R[t^{-1},It^w]} = R[t^{-1},q_1t^{p_1},\ldots,q_nt^{p_n}]$ which defines a Noetherian subvaluation, f. Hence we may construct $h_1,\ldots,h_s$ from f as in 3.35.

Suppose $g = h_i$ is a valuation for some i and consider the set $S := \{x \in R | g(x) = f(x)\}$. Since g is a valuation, S is f-compatible and is contained in a maximal f-compatible set, S_j. Since f is a subvalution, there is a valuation v_j with $v_j(x) = f(x)$ on S_j. Hence $g(x) = f(x)$ implies $v_j(x) = f(x)$. By construction $g(a_1),\ldots,g(a_m)$ are the unique integers such that $g(q_j) = \sum_{k=1}^m c_{jk}g(a_k)$ for $1 \le j \le n$, $g(x) \ge f(x)$ for all $x \in R$, and $g(y) = f(y)$ for any y in the specific size m subset of $\{a_1,\ldots,a_m,q_1,\ldots,q_n\}$ associated with g. Since v_j satisfies these properties, $v_j(a_k) = g(a_k)$ for $1 \le k \le m$. Now $\mathscr{R}(g) = R[t^{-1},a_1t^{g(a_1)},\ldots,a_mt^{g(a_m)}]$, so each generator of $\mathscr{R}(g)$ is an element of $\mathscr{R}(v_j)$. This means that $g(x) \le v_j(x)$ for all $x \in R$ so that $v_j(x) = f(x)$ implies $g(x) = f(x)$. Therefore, $S_j = S$.

If $g \ne v_j$, then there exists $x \in R$ with $f(x) \ne \min_{i \ne j}(v_i(x), g(x))$. In this case $g(x) < f(x)$, which contradicts the construction of g, or $f(x) = v_j(x) < g(x)$ which contradicts $S = S_j$. Therefore $g = v_j$ is a Rees valuation of I. □

The following example gives an ideal which is not described by previous results, but for which we may now conclude that all of its Rees valuations are Noetherian.

Example 3.39. $R = \mathbb{R}[x,y,z]$, $I = (x^3(y^2+z^2),(y^2+z^2)^2)$. We may not rewrite I as a monomial ideal by change of variables and for $b = (y^2+z^2)^2$ it is not true that $(I^2 : b) = I$. Hence neither of Theorems 3.30 or 3.32 demonstrate that the Rees valuations are Noetherian.

However, $\mathscr{R}(f) = \overline{R[t^{-1},x^3(y^2+z^2)t^6,(y^2+z^2)^2t^6]}$ and the irreducible factors of the generators

$x^3(y^2+z^2),(y^2+z^2)^2$ are $x,(y^2+z^2)$. These factors form a regular sequence in R and generate a prime ideal. Therefore, we may apply Lemma 3.34 to see that every filtration formed in Theorem 3.38 is a valuation.

Let $\beta=(y^2+z^2)$. We may calculate $\mathscr{R}(f)=R[t^{-1},\beta t^3,x\beta t^4,x^2\beta t^5,x^3\beta t^6]$. We apply Construction 3.35 with $a_1=x,a_2=\beta,q_1=x\beta,q_2=x^2\beta,q_3=x^3\beta$. We obtain two Noetherian Rees valuations $\mathscr{R}(v_1)=R[t^{-1},xt,\beta t^3]$ and $\mathscr{R}(v_2)=R[t^{-1},\beta t^6]$.

3.3 j-Multiplicities of Filtrations

Our next goal is to define j-multiplicity of Noetherian filtrations and to prove a result for Noetherian filtrations of maximal analytic spread which corresponds to Theorem 3.26. This follows naturally once we establish that this j-multiplicity is additive and is preserved under reductions of filtrations.

Let $(R,\mathfrak{m})$ be a local ring of Krull dimension d, I an ideal, and M a finitely generated R-module. We write $\Gamma(-)$ for the zeroth local cohomology functor $H^0_{\mathfrak{m}}(-)$. The multiplicity introduced in [1] may be defined as

$$j(I,M)=\lim_{n\to\infty}\frac{(d-1)!\lambda(\Gamma(I^{n-1}M/I^nM))}{n^{d-1}}$$

If we directly replace I^n with f_n from a Noetherian filtration, f, $\lambda\Gamma(f_{n-1}M/f_nM)$ defines a quasi-polynomial of degree at most $d-1$ as n gets large. As in the case of an $\mathfrak{m}$-primary filtration, taking $\sum_{i=0}^{n}\lambda\Gamma(f_{i-1}M/f_iM)$ produces a uniform quasi-polynomial in n of degree at most d. By the subadditivity of Γ, $\sum_{i=1}^{n}\lambda\Gamma(f_{i-1}M/f_iM)\geq\lambda\Gamma(M/f_nM)$, where we had equality when f_1 was $\mathfrak{m}$-primary. Analysis of the latter expression would seem to generalize the ε-multiplicity of Katz, Ulrich, Validashti and would be an interesting topic for future study.

In expounding the basic properties of j-multiplicity for filtrations, we follow the work of Flenner, O'Carroll and Vogel [11]. We first define a version of j-multiplicity for a graded ring which may not have a standard grading. Applying this to $G_f:=\mathscr{R}(f)/(t^{-1})\mathscr{R}(f)$ will give us the j-multiplicity of a filtration.

Let G be a graded ring with $G_0 = R$ which is finitely generated over $(R, \mathfrak{m})$, a local ring. Let M be a finitely generated graded G-module. We have that $\Gamma(M)$ embeds into $M/\mathfrak{m}^s M$ for sufficiently high s, so $\dim(\Gamma(M)) \leq \dim(M/\mathfrak{m}^s M)$. Since $\Gamma(M)$ is a graded module of $G/\mathfrak{m}^s G$, whose zeroth graded piece is an Artinian ring, $\lambda(\Gamma(M)_n)$ is eventually quasi-polynomial. We may obtain a uniform quasi-polynomial by taking $P(n) = \sum_{i=0}^{n} \lambda(\Gamma(M)_n)$.

Definition 3.40. Let $(R, \mathfrak{m})$ be a local ring with $\Gamma = H^0_{\mathfrak{m}}$. Let G be a graded ring which is finitely generated over $G_0 = R$ with $\dim(G) = d$. Let M be a graded G-module. Define $j(G, M)$ to be the rational number α for which $\frac{\alpha}{d!}$ is the common coefficient of n^d in the polynomials determining $\sum_{i=0}^{n} \lambda(\Gamma(M)_i)$.

$$j(G,M) = \lim_{n\to\infty} \frac{d!}{n^d} \sum_{i=0}^{n} \lambda(\Gamma(M)_i)$$

In contrast with the classical graded j-multiplicity, we remove the assumption that G is generated over G_0 in degree 1. Hence we have a quasi-polynomial instead of a polynomial function, and a rational 'multiplicity' instead of an integer.

Lemma 3.41. *Let $P(n)$ be a quasi-polynomial of period k and degree $d-1$. Let it be determined by the polynomials $P_0, \ldots, P_{k-1}$. Let $e_0, \ldots, e_{k-1}$ denote $(d-1)!$ times the coefficient of n^{d-1} in $P_0, \ldots, P_{k-1}$, respectively. Then the common leading term of the quasi-polynomial $Q(n) = \sum_{i=0}^{n} P(i)$ is $\frac{n^d}{k(d!)} \sum_{i=0}^{k-1} e_i$.*

Proof. Let $Q(n) = \sum_{i=0}^{n} P(i)$. Setting $n = qk + r$ for $r \leq k$, we have

$$Q(n) = \sum_{j=0}^{r} \sum_{i=0}^{q} P_j(ik+j) + \sum_{j=r+1}^{k-1} \sum_{i=0}^{q-1} P_j(ik+j)$$

We need only consider the leading term of $P_j(ik+j)$, $\frac{e_j}{(d-1)!}(ik+j)^{d-1}$. In fact, Lemma 2.20 asserts that the leading term of $\sum_{i=0}^{n} i^d$ is $n^{d+1}/(d+1)$. The only term, then, of the binomial expansion of $(ik+j)^{d-1}$ which will contribute to the leading term of $\frac{e_j}{(d-1)!} \sum_{i=0}^{\frac{n-r}{k}} (ik+j)^{d-1}$ is $(ik)^{d-1}$.

The leading term of $\frac{k^{d-1} e_j}{(d-1)!} \sum_{i=0}^{\frac{n-r}{k}} i^{d-1}$ is equal to $\frac{k^{d-1} e_j}{d!} \frac{n}{k}^d$. The same leading coefficient is obtained if we sum to $\frac{n-r}{k}-1$. Therefore, the leading coefficient of $Q(n)$ is $\sum_{i=0}^{k-1}\frac{e_j}{k(d!)}=\frac{1}{k\cdot(d!)}\sum_{i=0}^{k-1}e_j$. ☐

Example 3.21 is also an example of this lemma.

The additivity of the graded j-multiplicity in this setting follows along the same lines as that of Flenner, O'Carroll, Vogel. In their proof of [11] 6.1.2, the authors use neither the fact that $P(n)$ is a polynomial nor that G is standard graded.

Proposition 3.42. *Let G be a graded ring and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of graded G-modules. Then $j(M) = j(M') + j(M'')$.*

Proof. By the exact sequence, $\dim(\Gamma(M)) \geq \dim(\Gamma(M')), \dim(\Gamma(M''))$. Both sides of the desired equality are 0 whenever $\dim(G) > \dim(\Gamma(M))$. Now suppose $\dim(G) = \dim(\Gamma(M))$.

We begin by noting that $\Gamma(-)$ is left exact and claim that C, the cokernel of $\Gamma(M) \to \Gamma(M'')$, has dimension smaller than $\dim(\Gamma(M))$. Localization is exact and commutes with Γ, so we may consider the sequences $0 \to M'_P \to M_P \to M''_P \to 0$. For P a prime of dimension d, $\Gamma(M_P)$ is supported only on the local maximal ideal, so $\Gamma(M_P) = M_P$, $\Gamma(M'_P) = M'_P$, and $\Gamma(M''_P) = M''_P$. It follows that if $\dim(R/P) = d$, $0 \to \Gamma(M'_P) \to \Gamma(M_P) \to \Gamma(M''_P) \to 0$ so that $C_P = 0$. Hence $\dim(C) < d$.

We now consider the exact sequence

$$0 \to \Gamma(M') \to \Gamma(M) \to \Gamma(M'') \to C \to 0$$

Length is additive, so $\lambda(\Gamma(M)_i) + \lambda(C_i) = \lambda(\Gamma(M')_i) + \lambda(\Gamma(M'')_i)$ and for $n \gg 0$, these are quasi-polynomials. For the sums, $\sum_{i=0}^{n}\lambda(\Gamma(M)_i) + \sum_{i=0}^{n}\lambda(C_i) = \sum_{i=0}^{n}\lambda(\Gamma(M')_i) + \sum_{i=0}^{n}\lambda(\Gamma(M'')_i)$, Lemma 3.41 gives that each sum is determined by a uniform quasi-polynomial when $n \gg 0$. By adding the degree d terms of each quasi-polynomial, $j(M) + j(C) = j(M') + j(M'')$. Now $\dim(C) < d$ implies that $j(C) = 0$ and $j(M) = j(M') + j(M'')$. ☐

We now define the j-multiplicity of a filtration.

Definition 3.43. Let $(R,\mathfrak{m})$ be a local ring of Krull dimension d and let f be a Noetherian filtration on R. The Rees ring of an R-module, M, with respect to f is $\mathscr{R}_f(M) := \oplus_{i\in\mathbb{Z}} f_iM \subset M[t^{-1},t]$. The associated graded module is $G_f(M) := \mathscr{R}_f(M)/t^{-1}\mathscr{R}_f(M) = \oplus_{i=0}^{\infty}\frac{f_iM}{f_{i+1}M}$. We define $j(f,M)$ for a filtration f on R by $j(f,M) = j(G_f(M))$. We denote $j(f,R)$ by $j(f)$.

Recall that for an ideal I in a local ring R, $j(I) \neq 0$ if and only if I has maximal analytic spread. The analytic spread of f is defined by $\dim(G_f/\mathfrak{m}G_f) = \dim(\Gamma(G_f))$. If $\dim(R) = d$, then $j(f) = j(G_f) = \lim_{n\to\infty}\frac{d!}{n^d}\sum_{i=0}^{n}\lambda(\Gamma((G_f))_i) \neq 0$ if and only if $\dim(\Gamma(G_f)) = d$, in which case f has maximal analytic spread.

In order to analyze this multiplicity in the manner of the traditional j-multiplicity, we must prove a version of the Artin-Rees lemma for filtrations. This proof is a modified version of the proof in Matsumura ([27] Theorem 8.5).

Lemma 3.44. *Let f be a Noetherian filtration on a ring R. Let $N \subset M$ be finitely generated R-modules. There exists positive integers b,c such that for every $n > bc$,*

$$f_nM\cap N = \sum_{z=1}^{b} f_{n-bc-z}(f_{bc+z}M\cap N)$$

Proof. The Rees ring of f, $\mathscr{R}(f)$ is finitely generated over R by $a_1,\ldots,a_r$. Similarly, let M be generated by $m_1,\ldots,m_s$. Any $\alpha \in f_nM$ may be written as $\sum_{i=1}^{s} r_it^nm_i$ where $r_it^n \in \mathscr{R}(f)$. We consider $J_n = \{(r_1t^n,\ldots,r_st^n) \subset \mathscr{R}(f)_s | \sum_i r_it^nm_i \in N\}$. Finally, let $L \subset \mathscr{R}(f)_s$ be generated by $\cup_{i\geq 0}J_i$.

Since $\mathscr{R}(f)$ is Noetherian, L is a finitely generated R-submodule of $\mathscr{R}(f)_s$. Hence it is generated by a finite collection $u_1,\ldots,u_p$ with $u_j = (u_{j1},u_{j2},\ldots,u_{js}) \in J_{d_j}$. By Theorem 3.12, there exists a b with $f_{n+b} = f_nf_b$ for all $n > b$. Set $c = \max_j(\lfloor\frac{d_j}{b}\rfloor)$.

Let $n > bc$ and $\alpha \in f_nM\cap N$, $\alpha = \sum r_im_i$ with $(r_1t^n,\ldots,r_st^n) \in J_n$. We may write this as $(r_1t^n,\ldots,r_st^n) = \sum_j p_ju_j$ with $p_j \in \mathscr{R}(f)_{n-d_j}$. Now,

$$\alpha = \sum r_im_i = \sum_j p_j\sum_i u_{ij}m_i \in f_{n-d_j}(f_{d_j}M\cap N)$$

Since $bc > d_j$, there exists $z_j < b$ with $f_{n-d_j}(f_{d_j}M \cap N) = f_{n-bc-z_j}(f_{bc+z_j}M \cap N)$. Hence we conclude $\alpha \in \sum_{z=1}^{b} f_{n-bc-z}(f_{bc+z}M \cap N)$. $\square$

This modified Artin-Rees condition is sufficient to preserve the argument of [11] Lemma 1.2.6. This result bears the weight of our proof that the j-multiplicity of Noetherian filtrations is additive.

Lemma 3.45. *Let f be a Noetherian filtration on a ring R. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R modules. Then the modules $ker(G_f(M') \to G_f(M))$ and $ker(\frac{G_f(M)}{im(G_f(M'))} \to G_f(M''))$ have the the same minimal primes on $Spec(G_f(R))$.*

Proof. Let $\mathfrak{R}(M)$ be the Rees algebra of the module M and consider $K := ker(\mathfrak{R}(M) \to \mathfrak{R}(M''))$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(1) & \longrightarrow & \mathfrak{R}(M)(1) & \longrightarrow & \mathfrak{R}(M'')(1) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle t^{-1}} & & \Big\downarrow{\scriptstyle t^{-1}} & & \Big\downarrow{\scriptstyle t^{-1}} & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathfrak{R}(M) & \longrightarrow & \mathfrak{R}(M'') & \longrightarrow & 0 \end{array}$$

Since (t^{-1}) is a non-zero-divisor on $\mathscr{R}(M'')$, the snake lemma gives us a short exact sequence of cokernels.

$$0 \to K/t^{-1}K \to G(M) \to G(M'') \to 0$$

Let L be the cokernel of the inclusion map $\mathscr{R}(M') \hookrightarrow K$. Apply the snake lemma to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{R}(M')(1) & \longrightarrow & K(1) & \longrightarrow & L(1) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle t^{-1}} & & \Big\downarrow{\scriptstyle t^{-1}} & & \Big\downarrow{\scriptstyle t^{-1}} & & \\ 0 & \longrightarrow & \mathfrak{R}(M') & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

We obtain the exact sequence $0 \to U \to G(M') \to K/t^{-1}K \to V \to 0$

U is the cokernel of $L(1) \to L$ as well as $ker(G(M') \to G(M))$. V is the $ker(L(1) \to L)$ as well as $ker(G(M)/G(M') \to G(M''))$.

Observe that $K_n = M' \cap f_n M_1$ by definition, so $L = \oplus_n (M' \cap f_n M)/f_n M'$. Hence there exists a power a with $t^{-a}L = 0$ by Lemma 3.44. By the exact sequence

$$0 \to U \to L(1) \to L \to V \to 0$$

the minimal primes of L are also minimal primes of U and V. In addition, $\lambda(U_p) = \lambda(V_p)$ for all such primes. $\square$

Now we see that the j-multiplicity of filtrations is additive.

Theorem 3.46. *Let f be a Noetherian filtration on a local ring R. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then $j(f;M) = j(f;M') + j(f;M'')$.*

Proof. Let $N := K/t^{-1}K$ as in the previous proof. Recall from the previous proof the exact sequences

$$0 \to K/t^{-1}K \to G_f(M) \to G_f(M'') \to 0$$

and

$$0 \to U \to G_f(M') \to K/t^{-1}K \to V \to 0$$

By Proposition 3.42, the first exact sequence yields $j(G_f(M)) = j(G_f(M'')) + j(N)$. The second exact sequence gives $j(U) + j(N) = j(G_f(M')) + j(V)$. However, $j(U) = j(V)$ by 3.45, and we conclude that $j(N) = j(G_f(M'))$. Now, $j(G_f(M)) = j(G_f(M')) + j(G_f(M''))$ follows from the first equality.

Since $j(f;M) = j(G_f(M))$, we have the desired result. $\square$

While the proof of additivity is only a slight modification of the proof for the original j-multiplicity, it is better to follow a new line of thought when we turn to reductions of filtrations. For $g \leq f$ a reduction of filtrations, there exists z such that $g_z f_n = f_{n+z}$ for $n \gg 0$, but we must deal with a sequence of terms $g_z f_z, g_z f_{z+1}, \ldots, g_z f_{2z-1}$ before we increment the power of g_z. We avoid significant complications in the direct approach by relating the j-multiplicity of a Noetherian filtration to that of an ideal. The next proposition is a generalization of the fact that for I an ideal in a ring of Krull dimension d, $j(I^z) = z^d j(I)$ which may be found in [20] Corollary 3.11. In fact, this argument provides a new proof of that statement.

Proposition 3.47. *Let R be a local ring of dimension d and let f a Noetherian filtration on R. Fix $z \in \mathbb{N}$ and let g be the filtration defined by $g_i = f_{zi}$. Then $j(g) = z^d j(f)$.*

Proof. Consider the graded ring of g: $G_z = \oplus_{i=0}^{\infty} \frac{g_i}{g_{i+1}}$. This is isomorphic to $\oplus_{i=0}^{\infty} \frac{f_{zi}}{f_{(i+1)z}}$. Consider the following exact sequences.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+1}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi}}{f_{zi+1}} & \longrightarrow & 0 \\
0 & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+2}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+1}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+1}}{f_{zi+2}} & \longrightarrow & 0 \\
\vdots & & & & & & & & \\
0 & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+z-1}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+z-2}}{f_{z(i+1)}} & \longrightarrow & \oplus_{i=0}^{\infty} \frac{f_{zi+z-2}}{f_{zi+z-1}} & \longrightarrow & 0
\end{array}
$$

Denote $\oplus_{i=0}^{\infty} \frac{f_{zi+\ell}}{f_{zi+\ell+1}}$ by B_ℓ for each $0 \leq \ell \leq z-1$. These B_ℓ are graded modules over G_z, so each module has a well-defined j-multiplicity. Moreover, j-multiplicity is additive on exact sequences by Theorem 3.46, so $j(g) = \sum_{\ell=0}^{z-1} j(B_\ell)$.

Writing the latter by the limit definition, we have $j(g) = \sum_{\ell=0}^{z-1} \lim_{n_\ell \to \infty} \frac{d!}{n_\ell^d} \sum_{i=0}^{n_\ell - 1} \lambda \Gamma \left(\frac{f_{zi+\ell}}{f_{zi+\ell+1}} \right)$. Since each of these limits converges, we may regard the sum as a single limit in which the z-tuple $(n_0, \ldots, n_{z-1}) \to \infty$ along the path $n_0 = n_1 = \ldots = n_{z-1} = n$.

Now

$$
\begin{aligned}
j(g) &= \lim_{n \to \infty} \frac{d!}{n^d} \sum_{\ell=0}^{z-1} \sum_{i=0}^{n-1} \lambda \Gamma \left(\frac{f_{zi+\ell}}{f_{zi+\ell+1}} \right) \\
&= \lim_{n \to \infty} \frac{d!}{n^d} \sum_{s=0}^{nz-1} \lambda \Gamma \left(\frac{f_s}{f_{s+1}} \right) \\
&= z^d \lim_{n \to \infty} \frac{d!}{(nz)^d} \sum_{s=0}^{nz-1} \lambda \Gamma \left(\frac{f_s}{f_{s+1}} \right) = z^d j(f)
\end{aligned}
$$

□

Proposition 3.48. *Let $g \leq f$ be a reduction of Noetherian filtrations on a local ring R. Then $j(g) = j(f)$.*

Proof. By Proposition 3.14, there exists $z \in \mathbb{N}$ such that $g_{nz} = g_z^n$, $f_{nz} = f_z^n$ and g_z is an ideal reduction of f_z.

Apply Proposition 3.47 to f and g to get $j(g) = j(g_z)/z^d$ and $j(f) = j(f_z)/z^d$. Since $j(g_z) = j(f_z)$ as a reduction of ideals, we have the result. $\square$

For a valuation v on a ring R, recall that $v(I)$ denotes the $\min\{v(x) | x \in I\}$ for any ideal $I \subset R$. We may similarly define $v(f) := v(f_1)$. With this definition, we may relate $j(f)$ to $v_i(f)$ for $v_1, \ldots, v_s$ the Rees valuations of f, just as Katz and Validasthi related $j(I)$ to $v_i(I)$; [20]. This is one of the main results of this section.

Theorem 3.49. *Let f be a Noetherian filtration on a local ring R. Let $v_1, \ldots, v_s$ be the Rees valuations of f. There exist nonnegative rational numbers $d(f, v_i)$ such that*

$$j(f) = \sum_{i=1}^{s} d(f, v_i) v_i(f)$$

Proof. Let $I = f_z$ such that $f_{nz} = I^n$. By 1.42, there exist nonnegative integers $d(I, v_i)$ such that $j(I) = \sum_{i=1}^{s} d(I, v_i) v_i(I)$. Hence $j(f) = \frac{j(I)}{z^d} = \frac{1}{z^d} \sum_{i=1}^{s} d(I, v_i) v_i(I)$.

Now $v_i(f)$ is an integer with $v_i(f) \leq v_i(I)$, so $j(f) = \sum_{i=1}^{s} \frac{d(I,v_i)v_i(I)}{z^d v_i(f)} \cdot v_i(f)$. We may choose $d(f, v_i) = \frac{d(I,v_i)v_i(I)}{z^d v_i(f)}$. $\square$

The fact that $\frac{v_i(I)}{v_i(f)}$ is a factor of our constant is somewhat dissatisfying. However, it is not possible to give a more specific relationship between the formula for $j(f)$ and the formula for $j(I)$. Consider the following example.

Example 3.50. Let $R = k[x, y]$ and f the filtration for which $\mathscr{R}(f) = R[t^{-1}, xyt, x^3y^2t^3, x^2y^3t^3]$. f has a 3-repeating reduction g defined by $\mathscr{R}(g) = R[t^{-1}, x^3y^2t^3, x^2y^3t^3]$. One Rees valuation of f and g is the monomial valuation v with $v(x) = 1, v(y) = 0$. Then $v(f) = v(xy) = 1$ and $v(g) = v(x^3y^2 + x^2y^3) = 2$. Indeed for any $n \in \mathbb{N}$, we may define g_n as the minimal $3n$-repeating reduction of f and $v(g_n) = v((x^3y^2, x^2y^3)^n) = 2^n$.

Alternatively, we may define f' by $\mathscr{R}(f') = R[t^{-1}, x^2yt, x^3y^2t^3, x^2y^3t^3]$ and f' has the same reductions and Rees valuations, except $v(f') = 2$.

Chapter 4

Ideals of Submaximal Analytic Spread

In this chapter, we let $(R,\mathfrak{m})$ be a local ring of dimension d. We have defined one generalization of Hilbert-Samuel multiplicity by applying the functor $H^0_{\mathfrak{m}}$, denoted Γ, to identify a sequence of finite length modules.

$$\varepsilon(I) = \limsup_{n\to\infty} \frac{d!}{n^d}\lambda(\Gamma(\frac{R}{I^n}))$$

The study of such limits has be restricted to ideals of maximal analytic spread by the following result.

Theorem 4.1 ([42]; [20], Theorem 4.7)**.** *Let $(R,\mathfrak{m})$ be a local ring and let $I \subset R$ be an ideal. Then $\varepsilon(I) \neq 0$ if and only if I has maximal analytic spread.*

In this chapter, we begin to investigate the asymptotic behavior of $\lambda(\Gamma(R/I^n))$ for ideals with $\ell(I) < \dim(R)$. We find that for any such I, these lengths are eventually polynomial. When they are nonzero, the polynomial has degree at most $\ell(I)-1$. In Theorem 4.8, we show there exist ideals of any analytic spread t with $1 < t < \dim(R)$ for which $\lambda(\Gamma(R/I^n))$ is eventually polynomial of degree a for any $0 \le a < t-1$.

By definition $x \in \Gamma(R/I^n)$ if $x \in R/I^n$ and $\mathfrak{m}^k x = 0$ for some $k \in \mathbb{N}$. These are the images of elements from R which lie in $(I^n : \mathfrak{m}^k)$ for some $k \in \mathbb{N}$. Since $(I^n : \mathfrak{m}^k) \subset (I^n : \mathfrak{m}^{k+j})$ for any $j \in \mathbb{N}$, the $\cup_{k=0}^{\infty}(I^n : \mathfrak{m}^k)$ is eventually equal to the stable value of these ideals, which we denote by $(I^n : \mathfrak{m}^\infty)$. Thus, we shall frequently characterize $\Gamma(R/I^n)$ by $\frac{(I^n:\mathfrak{m}^\infty)}{I^n}$.

We find the following well-known proposition useful.

Proposition 4.2. $(I : \mathfrak{m}^\infty) \neq I$ *if and only if* $\mathfrak{m}$ *is an associated prime of* I.

Proof. Let $I \subset R$ be an ideal and consider the primary decomposition of $I = \cap_{i=1}^{t} p_i$. If p_1 is $\mathfrak{m}$-primary then $\cap_{i=2}^{t} p_i \subset (I : \mathfrak{m}^\infty)$.

Inversely, if $\mathfrak{m}$ is not an associated prime of I, then $\mathfrak{m}$ does not consist of zero divisors on R/I so $I = (I : \mathfrak{m}^\infty)$. $\square$

We have several results connecting analytic spread to associated primes.

Theorem 4.3 ([17], 5.4.6 and 5.4.7)**.** *If R is a Noetherian local ring and $\ell(I) = \dim(R)$, then $\mathfrak{m} \in \operatorname{Ass}(R/\overline{I^n})$ for all large n. If R is also quasi-unmixed, then $\mathfrak{m} \in \operatorname{Ass}(R/\overline{I^n})$ for some n if and only if $\ell(I) = \dim(R)$.*

In particular, if $\ell(I) < d$ and I is normal, then $\mathfrak{m}$ is not associated to any power of I and $\Gamma(R/I^n) = 0$ for all n.

A common construction in combinatorics is the edge ideal of a graph. Proposition 3.3 of [26] shows that for I, the edge ideal of a graph, $\mathfrak{m} \in Ass(R/I^k)$ for some k if and only if $\mathfrak{m} \in \operatorname{Ass}(R/\overline{I^t})$ for some t. Together with Theorem 4.3, this demonstrates that $\Gamma(R/I^n) = 0$ for all n if I is the edge ideal of a graph with $\ell(I) < \dim(R)$.

Some conditions sufficient to guarantee $\Gamma(R/I^n) = 0$ if and only if $\Gamma(R/I) = 0$ are given by Zarzuela ([44], Thm 3.1 + Thm 4.3) and Herzog, Rauf, Vladoiu ([15], Cor 4.5). For such ideals, $\Gamma(R/I^n) \neq 0$ for large n exactly when $\mathfrak{m} \in \operatorname{Ass}(I)$.

The following characterization of unmixedness given by Katz is particularly helpful. Recall that 'unmixed' is defined in 1.16.

Theorem 4.4 ([19], Theorem 4.1)**.** *Let $(R, \mathfrak{m})$ be a local ring. The following are equivalent:*

(i) R is unmixed.

(ii) For each ideal of the principal class I, with height $I < \dim R$, $\oplus_{n \geq 0}(I^n : \mathfrak{m}^\infty)$ is a finite $R[It]$-module.

(iii) For each ideal I with $\ell(I) < \dim(R)$, $\oplus_{n\geq 0}(I^n : \mathfrak{m}^\infty)$ is a finite $R[It]$-module.

By [9], if I contains a non-zero divisor, then $\mathrm{Ass}(I^n/I^{n-1}) = \mathrm{Ass}(R/I^n)$ for $n \gg 0$. Therefore, the sequences $\Gamma(I^n/I^{n+1})$ and $\Gamma(R/I^n)$ are non-vanishing for the same ideals. By (iii) of Theorem 4.4, we have $\oplus_{n\geq 0}(I^n : \mathfrak{m}^\infty)$ is finite over the standard graded R-algebra $\oplus_{n\geq 0} I^n$. By Theorem 1.5, $\lambda(\frac{(I^n:\mathfrak{m}^\infty)}{I^n})$ is eventually polynomial. Moreover, we have $(I^n : \mathfrak{m}^\infty) \subset I^{n-j}$. As was pointed out to me in conversation with Jack Jeffries and Youngsu Kim, we may combine this observation with the subadditivity of $\Gamma(-)$ to obtain

$$\lambda\Gamma(I^{n-1}/I^n) \leq \lambda\Gamma(R/I^n) \leq \lambda\Gamma(I^{n-1}/I^n) + \lambda\Gamma(I^{n-2}/I^{n-1}) + \ldots + \lambda\Gamma(I^{n-j}/I^{n-j+1})$$

For $n \gg 0$, $\lambda(\Gamma(I^{n-1}/I^n))$ is given by a polynomial, $P(n-1)$. Since this polynomial has a positive leading coefficient, we may assume

$$\begin{aligned}\lambda\Gamma(I^{n-1}/I^n) + \lambda\Gamma(I^{n-2}/I^{n-1}) + \ldots + \lambda\Gamma(I^{n-j}/I^{n-j+1}) &= \sum_{i=n-j}^{n-1} P(i) \\ &\leq jP(n-1) = j\lambda(\Gamma(I^{n-1}/I^n))\end{aligned}$$

Hence we obtain

$$\lambda\Gamma(I^{n-1}/I^n) \leq \lambda\Gamma(R/I^n) \leq j\lambda(\Gamma(I^{n-1}/I^n)) \text{ for } n \gg 0$$

Therefore the polynomials which eventually give $\lambda(\Gamma(I^{n-1}/I^n))$ and $\lambda(\Gamma(R/I^n))$ have the same degree. To find this degree, we consider the dimension of the A-module $H^0_J(A)$, for an arbitrary ring A.

Lemma 4.5. *Let A be a ring and $J \subset A$ an ideal.*

$$\dim(H^0_J(A)) = \max\left(\dim(A/Q) | J \subset Q \in \mathrm{Ass}(A)\right)$$

Proof. Let $P \in Supp(H^0_J(A))$ if and only if there exists $\alpha \neq 0$ in A_P which is annihilated by J_P.

Since $J_P \subset (0 :_{A_P} \alpha)$, $J_P \subset Q'$ for some $Q' \in \operatorname{Ass}(A_P)$. Lifting back to A, we find $Q \in \operatorname{Ass}(A)$ with $J \subset Q \subset P$.

Therefore, $Supp(H_J^0(A)) = \{P \in Spec(A) | \exists Q \in \operatorname{Ass}(A), J \subset Q \subset P\}$. $\square$

Letting $A = G(I) := \oplus_{n\geq 0} \frac{I^n}{I^{n+1}}$ and $J := \mathfrak{m}G(I)$, we find that the polynomials asymptotically defining $\lambda(H^0_{\mathfrak{m}}(I^n/I^{n+1}))$ and $\lambda((I^n : \mathfrak{m}^\infty)/I^n)$ have degree equal to the highest $\dim(G(I)/P) - 1$ such that $\mathfrak{m}G(I) \subset P \in \operatorname{Ass}(G(I))$. We are assuming that $\ell(I) = \dim(G(I)/\mathfrak{m}G(I)) < \dim(G(I))$, so the degree of the polynomial is given by an embedded associated prime of $G(I)$. Since our prime may not be minimal over $\mathfrak{m}G(I)$, the degree of growth may be strictly less than $\ell(I) - 1$.

In fact, such behavior may be observed. We give a theorem to show that the various combinations of analytic spread and degree of the growth of $\lambda\Gamma(R/I^n)$ which this indicates as possible will occur. We first construct an exact sequence which will aid in verifying the properties of our ideals.

Construction 4.6. Let $(R, \mathfrak{m})$ be an unmixed local ring of Krull dimension d. Let I be an ideal with $\ell(I) = t < d$ and suppose I has a minimal reduction $J = (g_1, \dots, g_t)$.

Since $\oplus_n (I^n : \mathfrak{m}^\infty)$ is a finite module over $R[It]$ which is a finite module over $R[Jt]$, there exists s such that $(I^{w+s} : \mathfrak{m}^\infty) = J^w(I^s : \mathfrak{m}^\infty)$ for all $w \in \mathbb{N}$. Since J is also a reduction of I, we may choose $z \in \mathbb{N}$ such that $\Gamma(R/I^n) = \frac{(I^n:\mathfrak{m}^\infty)}{I^n} = \frac{J^{n-z}(I^z:\mathfrak{m}^\infty)}{J^{n-z}I^z}$. Now let $U_1, \dots, U_t$ be new variables and consider the exact sequence:

$$0 \longrightarrow K \longrightarrow \frac{(I^z:\mathfrak{m}^\infty)[U_1,\dots,U_t]}{I^z[U_1,\dots,U_t]} \xrightarrow{\psi} \oplus_n \frac{J^n(I^z:\mathfrak{m}^\infty)}{J^n I^z} \longrightarrow 0 \tag{4.1}$$

where ψ is defined by taking $\psi(U_i) = g_i$. Note that ψ is a graded homomorphism

$$\left(\frac{(I^z : \mathfrak{m}^\infty)[U_1, \dots, U_t]}{I^z[U_1, \dots, U_t]}\right)_n \twoheadrightarrow \frac{J^n(I^z : \mathfrak{m}^\infty)}{J^n I^z}$$

The module on the left has length given by $\lambda(\frac{I^k:\mathfrak{m}^\infty}{I^k}) \cdot \binom{n+t-1}{t-1}$, a polynomial in n of degree $t-1$. Understanding $\ker(\psi)$ will tell us about the length we desire.

To help the reader digest the calculations which follow, it is helpful to consider an example.

Example 4.7. Let $R = k[x_1,\ldots,x_5]_{(x_1,\ldots,x_5)}$. Let $L = (x_1^2,x_2^2,x_3^2)+(x_4)(x_1,x_2,x_3,x_4)$ and define $I = L+\mathfrak{m}\overline{L}$. It is shown in the forthcoming proof that $(I^p : \mathfrak{m}^\infty) = (x_1,\ldots,x_4)^{2p}$ and we wish to characterize the monomials which lie in $(I^p : \mathfrak{m}^\infty)$ and outside $I^p = L^p + \mathfrak{m}L^{p-1}\overline{L} + \ldots + \mathfrak{m}^p\overline{L}^p$.

Let $p = 5$. Consider $\mu = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4} \in (I^5 : \mathfrak{m}^\infty)$ with $\sum_{i=1}^4 \alpha_i = 10$. First observe that if $\alpha_4 > 1$, then $\mu \in L^5 \subset I^5$. Suppose $\alpha_4 \geq 3$, then we may factor out x_ix_4 for any i with α_i odd; hence $\mu \in L^5$. If $\alpha_4 = 2$, then two or zero of $\alpha_1,\alpha_2,\alpha_3$ are odd and we may again conclude that $\mu \in L^5$.

Similarly, if $\alpha_4 = 1$ and exactly one of $\alpha_1,\alpha_2,\alpha_3$ is odd, then $\mu \in L^5$. If all three are also odd, then $\mu \notin L^5$. One may observe that by factoring out a single term $x_1x_2 \in \overline{L}\setminus L$, we find a monomial in L^4. Thus $\mathfrak{m}\cdot\mu \in \mathfrak{m}L^4\overline{L} \subset I^5$. Likewise, if $\alpha_4 = 0$, $\{\alpha_1,\alpha_2,\alpha_3\}$ may have 0 or 2 odd terms and μ falls into L^5 or $L^4\overline{L}$, respectively. From this one deduces that all elements of $(I^5 : \mathfrak{m}^\infty)/I^5$ have degree exactly 10 and so will have no factors of x^5.

Denote by $r(\mu)$ the number of exponents from $\alpha_1,\alpha_2,\alpha_3$ which are odd. We may summarize our results by saying: if $\mu \in (I^5 : \mathfrak{m}^\infty)$, then $\mu \notin I^5$ if and only if $\sum_{i=1}^5 \alpha_i - 10 < \frac{r(\mu)-\alpha_4}{2}$.

Theorem 4.8. *Let k be a field and let $k[x_1,\ldots,x_d]$ be the polynomial ring in d independent variables. Let R be the localization of $k[x_1,\ldots,x_d]$ at the homogeneous maximal ideal. There exists an ideal $I \subset R$ with analytic spread t and $\lambda(\Gamma(R/I^n))$ equal, for $n \gg 0$, to a polynomial of degree a for any $2 \leq t \leq d-1$ and any $0 \leq a \leq t-1$.*

Proof. Let $J = (x_1^2,\ldots,x_t^2)$. J is generated by a regular sequence of length t, so it has analytic spread t. Moreover, $\overline{J^n} = (x_1,\ldots,x_t)^{2n} = \overline{J}^n$ for all n. Each I we construct will have $J \subset I \subset \overline{J}$, so that J is a reduction of I and $\ell(I) = t$.

We investigate the kernel of ψ from the exact sequence described in equation 4.1

$$0 \longrightarrow K \longrightarrow \frac{(I^z:\mathfrak{m}^\infty)[U_1,\ldots,U_t]}{I^z[U_1,\ldots,U_t]} \xrightarrow{\psi} \oplus_n \frac{J^n(I^z:\mathfrak{m}^\infty)}{J^nI^z} \longrightarrow 0$$

where $\psi(rU_i) = rx_i^2$ for all $r \in \frac{(I^z:\mathfrak{m}^\infty)}{I^z}$. Since J^nI^z is a monomial ideal, the kernel of ψ will be a 'monomial' submodule in the variables $U_1,\ldots,U_t$. That is, a homogeneous form $f(U_1,\ldots,U_t)$

belongs to K if and only if each term $r\prod_{i=1}^{t} U_i^{q_i} \in K$.

In each of the following cases, we will pursue a numerical criterion on the exponents of a monomial $\mu \in (I^p : \mathfrak{m}^\infty)$ which will determine whether $\mu \in I^p$. Since this criterion behaves nicely under multiplication by x_i^2 for $1 \leq i \leq t$, this will give a characterization of K.

Case 1: $a = t-1$.

Let $I = J + \mathfrak{m}\overline{J}$.

$$I^p = J^p + \mathfrak{m}\overline{J}J^{p-1} + \ldots + \mathfrak{m}^p\overline{J}^p$$

Since $\overline{J} = (x_1, \ldots, x_t)^2$, one may quickly observe that $\overline{J}^p = \overline{J^p}$. Since $\ell(J) < d$, Theorem 4.3 tells us $(\overline{J^p} : \mathfrak{m}^\infty) = \overline{J^p}$. By the above description of I^p, we have $\mathfrak{m}^p\overline{J}^p \subset I^p$ so that $(\overline{J}^p : \mathfrak{m}^\infty) \subset (I^p : \mathfrak{m}^\infty)$. On the other hand, $I \subset \overline{J}$ so that $(I^p : \mathfrak{m}^\infty) \subset (\overline{J}^p : \mathfrak{m}^\infty) = \overline{J}^p$. Hence $\overline{J}^p = (I^p : \mathfrak{m}^\infty) = (x_1, \ldots, x_t)^p$.

Let $\mu = \prod_{i=1}^{d} x_i^{\alpha_i}$ be a monomial with $\sum_{i=1}^{t} \alpha_i \geq 2p$ so that $\mu \in (I^p : \mathfrak{m}^\infty)$. We will track three quantities which determine whether μ is also in I^p:

$$\delta_{p,t}(\mu) := \left(\sum_{i=1}^{t} \alpha_i\right) - 2p \qquad \delta_{p,d}(\mu) := \left(\sum_{i=1}^{d} \alpha_i\right) - 2p \qquad r(\mu) := \sum_{i=1}^{t} r_2(\alpha_i)$$

where $r_2(c) \in \{0,1\}$ denotes the residue of the integer c modulo 2. This last sum merely counts the number of exponents among the first t variables which are odd.

We first identify the largest number of factors of μ of the form x_i^2 with $1 \leq i \leq t$. This is given by $\frac{\sum_{i=1}^{t} \alpha_i - r(\mu)}{2}$, which is always an integer since $r_2(\sum_{i=1}^{t} \alpha_i) \cong \sum_{i=1}^{t} r_2(\alpha_i)$ modulo 2. Rewriting this in terms of $\delta_{p,t}(\mu)$ gives us $\frac{\sum_{i=1}^{t} \alpha_i - r(\mu)}{2} = \frac{\delta_{p,t}(\mu)+2p-r(\mu)}{2} = p - \frac{r(\mu)-\delta_{p,t}(\mu)}{2}$. Then $j = \frac{r(\mu)-\delta_{p,t}(\mu)}{2}$ is the smallest integer such that $\mu \in J^{p-j}\overline{J}^j$. Hence $\mu \in \mathfrak{m}^j J^{p-j}\overline{J}^j \subset I^p$ if $\delta_{p,d}(\mu) \geq j$. Conversely, if $\mu \in I^p$, then $\mu \in \mathfrak{m}^g J^{p-g}\overline{J}^g$ for some $0 < g < p$, so that μ has the described form.

$$\text{if } \mu \in (I^p : \mathfrak{m}^\infty), \text{ then } \mu \in I^p \text{ if and only if } \delta_{p,d}(\mu) \geq \frac{r(\mu) - \delta_{p,t}(\mu)}{2}$$

Now let $z \in \mathbb{N}$ be as in equation 4.1 and consider $\mu \in (I^z : \mathfrak{m}^\infty) \setminus I^z$. We wish to determine whether $\psi(\mu U_1^{b_1} \cdots U_t^{b_t}) = 0$. Denote $\sum_{i=1}^{t} b_i = B$ and $\mu \cdot \prod_{i=1}^{t} x_i^{2b_i} = \mu'$. We claim that $\mu' \notin I^{z+B}$.

We may calculate $\delta_{z+B,t}(\mu')$, $\delta_{z+B,d}(\mu')$, and $r(\mu')$ from $\delta_{z,t}(\mu), \delta_{z,d}(\mu)$ and $r(\mu)$. Observe

$$\begin{aligned}\delta_{z,t}(\mu) = \sum_{i=1}^{t} \alpha_i - 2z &= \sum_{i=1}^{t} \alpha_i + 2B - 2(z+B) \\ &= \sum_{i=1}^{t} (\alpha_i + 2b_i) - 2(z+B) = \delta_{z+B,t}(\mu')\end{aligned}$$

Similarly, one may see that $\delta_{z,d}(\mu) = \delta_{z+B,d}(\mu')$. Since all new factors are even powers of x_i, we also have $r(\mu) = r(\mu')$. Therefore if $\delta_{z,d}(\mu) < \frac{r(\mu)-\delta_{z,t}(\mu)}{2}$, then $\delta_{z+B,d}(\mu') < \frac{r(\mu')-\delta_{z+B,t}(\mu')}{2}$ and $\mu' \notin I^{z+B}$. Therefore, the kernel of ψ is zero.

For $n > z$, $\lambda(\Gamma(R/I^n)) = \lambda\left(J^{n-z}\frac{(I^z:\mathfrak{m}^\infty)}{J^{n-z}I^z}\right) = \lambda\left(\left(\frac{(I^z:\mathfrak{m}^\infty)[U_1,\dots,U_t]}{I^z[U_1,\dots,U_t]}\right)_{n-z}\right)$

This last length is $\left(\lambda\left(\frac{(I^z:\mathfrak{m}^\infty)}{I^z}\right)\cdot\binom{n-z+t-1}{t-1}\right)$ which is a polynomial in n of degree $t-1$, as promised.

Case 2: $1 \leq a < t-1$

Define a new ideal $L = (x_1^2, \dots, x_{a+1}^2) + (x_1, \dots, x_t)(x_{a+2}, \dots, x_t)$. Let our new $I := L + \mathfrak{m}\overline{L}$. Since $J \subset L \subset I \subset \overline{J}$, J is a reduction of I and L and we have the same exact sequence as in equation 4.1.

Set $\mu = \prod_{i=1}^{d} x_i^{\alpha_i}$. Let $\delta_{p,t}$, $\delta_{p,d}$ be as in Case 1 and redefine $r(\mu) = \sum_{i=1}^{a+1} r_2(\alpha_i)$: the number of odd exponents among the first $a+1$. Additionally, define $\sigma_{a,t}(\mu) = \sum_{i=a+2}^{t} \alpha_i$.

Since mixed terms involving $x_{a+2}, \dots, x_t$ lie in L, we have more flexibility in identifying generators of L which are factors of μ. We again seek the largest number of such factors. Whenever $\sigma_{a,t}(\mu) \geq r(\mu)$, each odd exponent from among the first $a+1$ terms may be factored out as a mixed term as we showed in Example 4.7. This demonstrates $\mu \in L^p \subset I^p$. If $r(\mu) > \sigma_{a,t}(\mu)$, then we have at most $\frac{1}{2}(\sum_{i=1}^{t} \alpha_i - r(\mu) + \sigma_{a,t})$ factors. Simplifying, we find

$$\begin{aligned}\frac{1}{2}(\sum_{i=1}^{t} \alpha_i - r(\mu) + \sigma_{a,t}(\mu)) &= \frac{1}{2}(2p + \delta_{p,t}(\mu) - r(\mu) + \sigma_{a,t}(\mu)) \\ &= p - \frac{1}{2}(r(\mu) - \delta_{p,t}(\mu) - \sigma_{a,t}(\mu)) = p - j\end{aligned}$$

Taking this last equation as the definition of j, j is the least integer such that $\mu \in L^{p-j}\overline{L}^j$. In this case, $\mu \in I^p$ if $\delta_{p,d}(\mu) \geq j$. We again have the converse by observing that $\mu \in I^p$ implies

$\mu \in \mathfrak{m}^g L^{p-g}\overline{L}^g$ for some $0 \le g \le p$.

$$\text{if } \mu \in (I^p : \mathfrak{m}^\infty), \text{then } \mu \in I^p \text{ if and only if } \delta_{p,d} \ge \frac{1}{2}(r(\mu) - \delta_{p,t}(\mu) - \sigma_{a,t}(\mu))$$

Note that by taking the residues modulo 2, we have

$$\begin{aligned} r_2\left(r(\mu) - \delta_{p,t}(\mu) - \sigma_{a,t}(\mu)\right) &= r_2\left(\sum_{i=1}^{a+1} r_2(\alpha_i) - \sum_{i=1}^{t} r_2(\alpha_i) - \sum_{i=a+2}^{t} r_2(\alpha_i)\right) \\ &= r_2\left(-2\sum_{i=a+2}^{t} r_2(\alpha_i)\right) \end{aligned}$$

so that the right side of the inequality is always an integer.

We now wish to apply this criterion to analysing the kernel of ψ. Again let $\mu \in (I^z : \mathfrak{m}^\infty) \setminus I^z$ and consider $\psi(\mu \cdot \prod_{i=1}^t U_i^{b_i}) = \mu'$. As in case 1, we have $\delta_{z+B,d}(\mu') = \delta_{z,d}(\mu)$, $\delta_{z+B,t}(\mu') = \delta_{z,t}(\mu)$ and $r(\mu') = r(\mu)$. The only term of our criterion which may change is $\sigma_{a,t}$.

If $b_{a+2} = b_{a+3} = \ldots = b_t = 0$, then $\sigma_{a,t}(\mu) = \sigma_{a,t}(\mu')$ and $\mu' \notin I^{z+B}$. This indicates that the submodule generated by $(U_1, \ldots, U_{a+1})$ lies entirely outside K. There are $\binom{n+a}{a}$ monomials in $U_1, \ldots, U_{a+1}$ of degree n, so $\lambda((I^z : \mathfrak{m}^\infty)/I^z) \cdot \binom{n-a}{a} \le \lambda((I^{z+n} : \mathfrak{m}^\infty)/I^{n+z})$.

On the other hand, we note that if $b_i \ne 0$ for any $a+1 < i$, then $\sigma_{a,t}(\mu) < \sigma_{a,t}(\mu')$ while all the other terms remain the same. Therefore, there exists w such that if $\prod_{i=1}^t U_i^{b_i} \in (U_{a+2}, \ldots, U_t)^w$, then $\psi(\mu \cdot \prod_{i=1}^t U_i^{b_i}) = 0$ for all $\mu \in (I^z : \mathfrak{m}^\infty)$.

Consider the following sum, which represents an upper bound for the number of monomials in U_i of degree n (for $n \gg w$) which might have a nonzero image for some coefficient μ.

$$\sum_{b_t=0}^{w} \sum_{b_{t-1}=0}^{w} \cdots \sum_{b_{a+2}=0}^{w} \binom{n-B+a}{a} < (w+1)^{t-a-2}\binom{n+a}{a}$$

This yields an upper bound on the length of interest

$$\lambda\left(\frac{(I^{n+z} : \mathfrak{m}^\infty)}{I^{n+z}}\right) < \lambda((I^z : \mathfrak{m}^\infty)/I^z)(w+1)^{t-a-2}\binom{n+a}{a}$$

Since we have an upper bound and a lower bound of our length which are polynomials of degree a and $\lambda(\frac{(I^n:\mathfrak{m}^\infty)}{I^n})$ is a polynomial, we conclude that $\lambda(\frac{(I^n:\mathfrak{m}^\infty)}{I^n})$ is a polynomial of degree a in n.

Case 3: $a = 0$

Let L be generated by all monomials in the first $t < d$ variables of total degree 3, except $x_1^2x_2$. Again set $I = L + \mathfrak{m}\overline{L} = L + (x_1^2x_2)(x_{t+1}, \dots, x_d)$, so that $\lambda((I:\mathfrak{m}^\infty)/I) = 1$. $J = (x_1^3, x_2^3, \dots, x_t^3)$ is a minimal reduction of L and I; $\overline{J} = (x_1, \dots, x_t)^3$.

For any n, one may see that $x_1^{3n-1}x_2 \in (I^n : \mathfrak{m}^\infty) \setminus I^n)$. However, for any other monomial of total degree $3n$ in the first t variables, we may factor out all copies of x_2 into terms of degree 3 without using $x_1^2x_2$. Such monomials lie in I^n.

Therefore, $\lambda(I^n : \mathfrak{m}^\infty)/I^n) = 1$ for all n. □

Further exploration is needed on the applications of these lengths. The first fact that is clear is that we will not have a Rees-type characterization of integral closures using the leading terms of these polynomials. The ideals in cases 1 and 2 of Proposition 4.8 demonstrate that we may change the degree of the polynomial without changing the integral closure of the ideal. We may even change the leading coefficient while preserving the degree of the polynomial and the integral closure of the ideals.

Example 4.9. Let $J_1 = (x^4, y^4) \subset J_2 = (x^4, x^2y^2, y^4)$. $I_1 = J_1 + \mathfrak{m}\overline{J_1}$ and $I_2 = J_2 + \mathfrak{m}\overline{J_2}$. Then $\lambda(I_1^n : \mathfrak{m}^\infty / I_1^n)$ and $\lambda(I_2^n : \mathfrak{m}^\infty / I_2^n)$ are given by polynomials $4n-1$ and $2n$ respectively. Although the leading coefficients are different, $\overline{I_1} = \overline{I_2}$.

I hope to expand on this work by comparing the leading coefficients of the polynomials giving these lengths to results concerning the secondary Hilbert coefficients of $\mathfrak{m}$-primary ideals. For example, given a Hilbert-Samuel polynomial in n of degree d, the coefficient of n^{d-1} is called the Chern number and has been well-studied, as in [25].

Chapter 5

Affine Semigroup Rings of Dimension 2

This chapter consists of joint work with Tony Se. We study monomial rings of the form

$$R = k[x^a, x^{p_1}y^{s_1}, \ldots, x^{p_t}y^{s_t}, y^b]$$

The principal motivation is the question of identifying when such rings are Cohen-Macaulay (or CM).

An important special case of this problem is given by $k[x^n, x^{n-a_1}y^{a_1}, \ldots, x^{n-a_t}y^{a_t}, y^n]$ for integers $0 < a_1 < \ldots < a_t < n$. These are projective monomial curves and the descriptor 'Cohen-Macaulay' arises in the context of their study. For example, the following is a famous example of Macaulay [24, p. 98].

Example 5.1. Let $R = k[x^4, x^3y, xy^3, y^4]$. The ring has dimension 2 since $0 \subset (x^4, x^3y, xy^3) \subset (x^4, x^3y, xy^3, y^4)$ are all prime ideals. However, R contains no regular sequence of length 2. For example, if $I = (x^4)$, then $(x^3y)^2 \in R \setminus I$ since $x^2y^2 \notin R$. However, $x^6y^2 \cdot y^4 \in (x^4)$, so y^4 is a zero divisor in $R/(x^4)$. Hence every element of $R/(x^4)$ is a unit or a zero divisor.

Alternatively, Macaulay and others noticed that many 'nice' rings have the property that for an $\mathfrak{m}$-primary ideal I with the least number of generators for being $\mathfrak{m}$-primary, we have $\lambda(R/I) = e(I)$. In R, a system of parameters is $I = (x^4, y^4)$, but $e(I) = 4$ and $\lambda(R/(x^4, y^4)) = 5$.

An essential manner of viewing this problem is by considering the semigroup of monomials which lie in the ring. For a polynomial ring with n variables, we let each monomial m be a point in $\mathbb{Z}^n$ corresponding to its exponent vector $\log(m)$. These points form a semigroup inside $\mathbb{Z}^n$ whose generators correspond to the monomials generating R over k.

One of the most important breakthroughs in the study of the CM property of these rings was made by Hochster.

Theorem 5.2 ([16, Theorem 1])**.** *If M is a monomial semigroup in the variables $x_1,\ldots,x_n$ and $k[M] \subset k[x_1,\ldots,x_n]$ is normal, then $R[M] \subset R[x_1,\ldots,x_n]$ is CM for any CM ring R.*

While this settles a great number of cases, there are plenty of monomial semigroups for which $k[M]$ is not normal. In particular, the projective monomial curves described in the first paragraph are never normal unless $t = n-1$. To see this, note that we may assume $\gcd(a_1,\ldots,a_t,n) = 1$, so that $\frac{x}{y}$ is in the fraction field of R. Each monomial of the form $x^i y^{n-i}$ is in the fraction field of R and satisfies the equation of integral dependence $(x^i y^{n-i})^n - (x^n)^i (y^n)^{n-i} = 0$. In a similar way, many rings of the form $k[x^a, x^{p_1}y^{s_1},\ldots,x^{p_t}y^{s_t},y^b]$ are not normal. For example, the fraction field of $R = k[x^2, x^{11}y, xy^{11}, y^3]$ contains $\frac{x^{11}y}{(x^2)^5} = xy$ and $(xy)^6 \in R$.

In the case of simplicial affine semigroups, Goto, Suzuki and Watanabe give another criterion by which to evaluate CM. A semigroup is affine if it may be embedded in $\mathbb{Z}^n$ for some n. For any affine semigroup, one may consider the cone: $C(S) = \{\alpha \in \mathbb{R}^n | k \cdot \alpha \in S \text{ for some } 0 \leq k \in \mathbb{R}\}$. The semigroup is said to be simplicial if the cone may be generated in $\mathbb{R}^n$ by $rank(S)$-many linearly independent elements of S as $\mathbb{R}^n$ vectors. Since rational cones of rank 2 always have two generators, our semigroups will be simplicial.

Theorem 5.3 ([12, Theorem 5.1]; [38, Theorem 6.4])**.** *Let S be a simplicial affine semigroup. Let $e_1,\ldots,e_s$ be elements which span C_S. Then $k[S]$ is CM if and only if*

$$\{x \in G | x + e_i \in S \text{ and } x + e_j \in S \text{ for some } i \neq j\} = S$$

Goto and Watanabe defined a similar extension S' for a general affine semigroup S and showed that $S' = S$ is necessary for $k[S]$ to be CM. Later, Trung and Hoa [40, Theorem 4.1] identify a topological criterion which, together with $S = S'$, is necessary and sufficient for CM.

The criterion of Theorem 5.3 is straightforward to check for a single ring, but does not lend itself to analysing classes of rings. Reid and Roberts [32] introduce a related notion of a maximal projective monomial curve in order to demonstrate a large class of CM curves. The special case of projective monomial curves continues to be studied.

In considering affine semigroup rings in dimension 2, our work emphasizes the congruence classes of the exponent vectors. This allows us to calculate the Hilbert polynomial of (x^a, y^b) in section 5.1. We continue using these classes to construct a basis for $R/(x^a, y^b)$ over k. The multiplicity of (x^a, y^b) and $\dim_k(R/(x^a, y^b))$ come together to form a classical criterion for the CM property.

Theorem 5.4 ([27, Theorem 17.11])**.** *Let $(R, \mathfrak{m})$ be a local ring. The following are equivalent:*

(i) *R is a Cohen-Macaulay ring.*

(ii) *$\lambda(R/I) = e(I)$ for every I generated by a system of parameters.*

(iii) *$\lambda(R/I) = e(I)$ for some I generated by a system of parameters.*

5.1 Asymptotic Behavior of the System of Parameters

For this section $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$ and the system of parameters of interest is x^a, y^b. In the following discussion, we assign x to have weight b and y to have weight a, so that $x^\alpha y^\beta$ will be said to have degree $b\alpha + a\beta$. This is so that x^a and y^b will have equal degree. When we wish to specifically highlight the exponent vector, we will use log: $\log(x^\alpha y^\beta) = (\alpha, \beta) \in \mathbb{Z}^2$.

Remark 5.5. There exists a ring isomorphism $\phi : R \to R'$ in which $\phi(x) = x^b$, $\phi(y) = y^a$ and $R' = k[x^{ab}, x^{bp_1}y^{aq_1}, \ldots, y^{ab}]$. Without loss of generality, we might have assumed that $a = b$. On the other hand, for $a \neq b$ we may freely assume $\gcd(a, p_1, p_2, \ldots, p_t) = \gcd(b, q_1, \ldots, q_t) = 1$.

Notation 5.6. As we consider $(x^a, y^b)^n$, we find it convenient to denote $X := x^a$ and $Y := y^b$.

Let $H \subset (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$ be the subgroup of residues which are the image of $log(\alpha)$ for some monomial $\alpha \in R$. For each $(p,q) \in H$, we choose $\alpha_{p,q}$ to be one monomial of minimal (weighted) degree in $R \setminus (X,Y)$. We will approximate $\lambda\left(\frac{(X,Y)^n}{(X,Y)^{n+1}}\right)$ by the number of elements in

$$A_n := \cup_{(p,q)\in H}\{\alpha_{p,q}X^n, \alpha_{p,q}X^{n-1}Y, \ldots, \alpha_{p,q}Y^n\}$$

The set of monomials of $(X,Y)^n$ outside of both $(X,Y)^{n+1}$ and A_n will be denoted B_n.

Set $\alpha_{p,q} = x^{\ell a+p}y^{mb+q} = {}_0\beta = \beta_0$. For $i < n$, let ${}_i\beta := x^{(\ell-i)a+p}y^{(m+j)b+q}$ with j the least possible integer such that ${}_i\beta \in R$. It may be that there is no such monomial, in which case we do not consider ${}_i\beta$ to be defined. Similarly, $\beta_i := x^{(\ell+j)a+p}y^{(m-i)b+q}$.

Example 5.7. Let $R = k[x^2, x^7y, xy^8, y^3]$. In this case, $H = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$. We may identify minimal elements $x^7y = \alpha_{1,1}$ and $xy^8 = \alpha_{1,-1}$. Since $x^7y \cdot xy^8 \in (x^2, y^3)$, the only monomials of R outside (x^2,y^3) are the first 5 powers of x^7y and xy^8. These are $x^{14}y^2, x^{21}y^3, x^{28}y^4, x^{35}y^5$ and $x^2y^{16}, x^3y^{24}, x^4y^{32}, x^5y^{40}$. Since $(7,1) \equiv (5,40)$, we write $x^5y^{40} = {}_{1,1,1}\beta$. Similarly, $(14,2) \equiv (4,32)$ and we write $x^{14}y^2 = \alpha_{1,2}$ and $x^4y^{32} = {}_{0,2,5}\beta$, etc.

We need to distinguish between monomials of the form $\beta_i X^i Y^{n-i}$ which lie in $(X,Y)^n$ and those which do not, so that we may count $|B_n|$.

Lemma 5.8. *Let $i > j$. If $\deg(\beta_i) \leq \deg(\beta_h)$ for all $h > j$ such that β_h is well-defined, then $\beta_i X^{n-i+h}Y^{i-h} \notin (X,Y)^{n+1}$ for $i \geq h > j$.*

Proof. If $\beta_i X^{n-i+h}Y^{i-h} \in (X,Y)^{n+1}$, then there exists $\gamma \in R$ with $\gamma X^{n+1-c}Y^c = \beta_i X^{n-i+h}Y^{i-h}$ for some $c \in \mathbb{N}$. By the minimality of the β's, β_{h+c} divides γ, so that $\deg(\beta_{h+c}) < \deg(\beta_i)$. $\square$

Lemma 5.9. *Fix $(p,q) \in H$. Let $s_{p,q} = s$ be the highest integer such that β_s is defined. Let u be the maximum value of $i - j - 1$, $s \geq i > j \geq 0$ such that $\deg(\beta_i) < \deg(\beta_h)$ for all $i > h > j$. Let $\mathscr{U}_{p,q,n} = \mathscr{U}_n = \{\beta_i X^{n-c}Y^c | 0 < i \leq s, 0 \leq c \leq n\}$. Then $|B_n \cap \mathscr{U}_n| \leq s$ with equality if and only if $n \geq u$.*

Proof. We first claim $|B_n \cap \mathscr{U}_n| \leq s$. By definition, $B_n \subset (X,Y)^n \setminus (X,Y)^{n+1}$ which means at most one monomial in $\mathscr{U}_n$ of a given y-exponent may lie in B_n. If $c \geq i$, then $\alpha_{p,q}X^{n-c+i}Y^{c-i}$ divides

$\beta_i X^{n-c}Y^c$, so $\beta_i X^{n-c}Y^c$ cannot be in B_n. There are only s-many other y-exponents the monomials in $\mathscr{U}_n$ might take.

Let $s = i_v > i_{v-1} > \ldots > i_0 = 0$ be integers such that $\deg(\beta_{i_v}) \geq \deg(\beta_{i_{v-1}}) \geq \ldots \geq \deg(\beta_0)$ and $\deg(\beta_{i_z}) < \deg(\beta_h)$ for all $i_z > h > i_{z-1}$.

Note that each pair i_z, i_{z-1} satisfies the defining condition of u. Moreover, $\deg(\beta_{i_z}) \leq \deg(\beta_h)$ for any $h > i_z$, so any pair i, j with $i > i_z > j$ fails the condition that defines u. Therefore, we find $u = \max_z(i_z - i_{z-1} - 1)$.

Suppose $n \geq u$ and consider the following monomials:

$$\begin{aligned}
&\beta_s X^n, \beta_s X^{n-1}Y, \ldots, \beta_s X^{n-(s-i_{v-1}-1)}Y^{s-i_{v-1}-1},\\
&\beta_{i_{v-1}}X^n, \ldots, \beta_{i_{v-1}}X^{n-(i_{v-1}-i_{v-2}-1)}Y^{i_{v-1}-i_{v-2}-1},\\
&\ldots,\\
&\beta_{i_1}X^n, \ldots, \beta_{i_1}X^{n-(i_1-1)}Y^{i_1-1}
\end{aligned}$$

By Lemma 5.8, each of these monomials lies outside $(X,Y)^{n+1}$, so $|B_n \cap \mathscr{U}_n| = s$.

Suppose $n < u$ and let i, j be indices satisfying $i - j - 1 = u$ and $\deg(\beta_i) < \deg(\beta_h)$ for all $i > h > j$. Consider the set of monomials $\beta_h X^{i-h-1}Y^{h-i+n+1}$ with $i > h \geq i-n-1$. By the assumption on n, $\deg(\beta_i) < \deg(\beta_h)$ for $i > h \geq i-n-1$ so $\beta_i Y^{n+1}$ divides each $\beta_h X^{i-h-1}Y^{h-i+n+1}$. Moreover, these $\beta_h X^{i-h-1}Y^{h-i+n+1}$ are the only monomials in $\mathscr{U}_n$ with the same y-exponent as $\beta_i Y^{n+1}$. Hence $B_n \cap \mathscr{U}_n$ does not contain a monomial with this y-exponent and $|B_n \cap \mathscr{U}_n| < s$. $\square$

Symmetry allows us to apply this result to ${}_t\beta, \ldots, {}_0\beta$. Define u' for ${}_t\beta, \ldots, {}_0\beta$ as u is defined for $\beta_s, \ldots, \beta_0$. For $n \geq n_{p,q} := \max(u, u')$, this yields $|B_n \cap \{\gamma | \log(\gamma) \equiv (p,q) \in H\}| = s + t$.

Proposition 5.10. *For any $n > 0$, $|B_n| \leq \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$. Moreover, equality holds if and only if $n \geq \max_{(p,q)\in H}(n_{p,q})$.*

Proof. We characterize B_n as the disjoint union $\cup_{(p,q)\in H}\left(\mathscr{U}_{p,q,n} \cap B_n\right)$. Now Lemma 5.9 shows

$n \geq \max_{(p,q)\in H} n_{p,q}$ if and only if

$$|\cup_{(p,q)\in H}(\mathscr{U}_{p,q,n}\cap B_n)| = \sum_{(p,q)\in H}|\mathscr{U}_{p,q,n}\cap B_n| = \sum_{(p,q)\in H}(t_{p,q}+s_{p,q})$$

$\square$

Not only is $|B_n|$ constant for large values of n, it determines the constant of the Hilbert polynomial $P(n) = \lambda((X,Y)^n/(X,Y)^{n+1})$ for $n \gg 0$. We demonstrate this by calculating the multiplicity from the growth of $|A_n|$.

Theorem 5.11. *Let $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$. The Hilbert polynomial of (x^a, y^b) is $P(n) = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q}+s_{p,q})$ where $H \subset (\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})$ is the subgroup generated by $(p_1,q_1),(p_2,q_2),\ldots,(p_t,q_t)$. In particular, $e((x^a,y^b)) = |H|$, and the Hilbert function equals the Hilbert polynomial for $n \geq \max_{(p,q)\in H}(n_{p,q})$.*

Proof. Applying the notation from 5.6, we have $P(n) = |A_n| + |B_n|$.

Since $A_n = \cup_{(p,q)\in H}\{\alpha_{p,q}X^{n-i}Y^i | 0 \leq i \leq n\}$, we have $|A_n| = |H|(n+1)$ for any n. With Lemma 5.9, we have $P(n) = |A_n| + |B_n| = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q}+s_{p,q})$ for all $n \geq \max_{(p,q)\in H}(n_{p,q})$.

$\square$

Example 5.12. Returning to example 5.7, we may identify the Hilbert polynomial as follows. Note that for each $\alpha_{p,q}$, either ${}_i\beta$ is undefined for all $i > 0$ or β_i is undefined for all β_i. In fact, the sequence $i_0, \ldots, i_v$ has $v = 0$ or $v = 1$ for every $(p,q) \in (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$.

(p,q)	$\alpha_{p,q}$	β	$i_1 = t_{p,q}+1$ or $s_{p,q}+1$
$(0,0)$	1		0
$(1,1)$	x^7y	x^5y^{40}	1
$(0,2)$	$x^{14}y^2$	x^4y^{32}	5
$(1,0)$	x^3y^{24}	$x^{21}y^3$	7
$(0,1)$	x^2y^{16}	$x^{28}y^4$	4
$(1,2)$	xy^8	$x^{35}y^5$	1

Thus, the Hilbert polynomial of $R = k[x^2, x^7y, xy^8, y^3]$ is $P(n) = 6n + 6 + 18$ and is equal to the Hilbert function if and only if $n \geq 6$.

Taken together, Theorem 5.11 and Proposition 5.10 allow us to construct rings with arbitrary conditions on the Hilbert polynomial and the level of its stabilization.

Corollary 5.13. *Given any subgroup $0 \neq H \subset (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$ and integers $C, m \geq 0$, there exists R such that (x^a, y^b) has Hilbert polynomial $P(n) = |H|(n+1) + C$ which equals the Hilbert function exactly for $n \geq m$.*

Proof. Fix $(0,0) \neq (p_0, q_0) \in H$ and an integer $N \geq C + m + 1$. For any $(0,0) \neq (p,q) \in H$ we let $\alpha_{p,q} = x^p y^q X^N Y^{N+C+m}$. Let $\beta_{p_0,q_0,j} = x^{p_0} y^{q_0} X^{N+j} Y^{N+C+m-j}$ for $j = 0, 1, 2, \ldots, C-1, C+m$. Let $S' = \{\alpha_{p,q} \mid (0,0) \neq (p,q) \in H\} \cup \{\beta_{p_0,q_0,j} \mid j = 0, 1, 2, \ldots, C-1, C+m\}$, $S = S' \cup \{x^a, y^b\}$ and $R = k[S]$. We will show that $(x^a, y^b) \subset R$ has the required Hilbert polynomial.

Let $\gamma, \delta \in S'$, and let $\log_x, \log_y$ denote the respective exponents of a monomial. We have $\log_x(\gamma\delta) \geq 2Na \geq (N+C+m+1)a > (N+C+m)a + p$ and $\log_y(\gamma\delta) \geq 2Nb > (N+C+m)b + q$ for any $0 \leq p < a$ and $0 \leq q < b$. In particular, $\log_x(\gamma\delta) > \log_x(\alpha_{p,q})$ and $\log_y(\gamma\delta) > \log_y(\alpha_{p,q})$ for any $\alpha_{p,q}$. Therefore, there is no ${}_{p,q,i}\beta$ for any $(p,q) \in H$ or $\beta_{p,q,j}$ for any $(p,q) \neq (p_0, q_0)$ or for $(p,q) = (p_0, q_0)$ with $j > C+m$. Similarly, $\log_x(\gamma\delta) > \log_x(\beta_{p_0,q_0,j})$ and $\log_y(\gamma\delta) > \log_y(\beta_{p_0,q_0,j})$ for any $j = 0, 1, 2, \ldots, C-1, C+m$.

Thus, $\deg(\beta_{p_0,q_0,j}) > \deg(\beta_{p_0,q_0,C+m})$ for all $j = C, C+1, \ldots, C+m-1$, and, in particular, $\beta_{p_0,q_0,j} = \beta_{p_0,q_0,C+m} Y^{C+m-j}$.

The maximum u as in Lemma 5.9 is $(C+m) - (C-1) - 1 = m$. Therefore $(x^a, y^b) \subset R$ has Hilbert polynomial $P(n) = |H|(n+1) + C$ which equals the Hilbert function exactly at $n \geq m$ by Proposition 5.10. □

Let us return to consideration of the CM property. In general, $\lambda(R/(X,Y)) \geq e((X,Y))$ and equality implies CM by 5.4.

Proposition 5.14. *The following are equivalent:*

(i) *R is CM*

(ii) $B_n = \emptyset$ *for all n.*

(iii) $B_i = \emptyset$ *for some i.*

Proof. $(i) \Rightarrow (ii)$ If R is CM, then by Theorems 5.4 and 5.11, $\lambda(R/(X,Y)) = |H|$. Then every monomial of R may be written as $\alpha_{p,q}X^iY^j$ for some $i, j \in \mathbb{N}$. Hence $B_n = \emptyset$ for all n.

$(iii) \Rightarrow (i)$ If R is not CM, then $\lambda(R/(X,Y)) > |H|$. By the pigeonhole principle, some congruence class $(p,q) \in H$ must be associated with two monomials in $R \setminus (X,Y)$. That is, for some $(p,q) \in H$, there is ${}_i\beta$ or β_j. If $s > 0$ is the highest integer such that β_s is defined, then $\beta_s X^i \in B_i$ for all i. $\square$

An alternative manner of viewing this result helps to motivate the calculations in the following sections. Impose the reverse lexicographic order on monomials in R. Let $\mu_{p,q}$ be the least monomial in this order such that $\log(\mu_{p,q}) \equiv (p,q)$. We use $\mathscr{B}_0$ to indicate the collection of $\mu_{p,q}$ for all $(p,q) \in H$. Alternatively, we may use the lexicographic order of the monomials, and form a set $\mathscr{B}'_0$ of elements $\mu'_{p,q}$.

Proposition 5.15. *The following are equivalent:*

(i) *R is CM*

(ii) $\mathscr{B}_0 = \mathscr{B}'_0$

(iii) $\mu_{p,q} = \mu'_{p,q}$ *for all* $(p,q) \in H$.

Proof. $(i) \Rightarrow (ii)$: If R is CM, then $\lambda(R/(x^a,y^b)) = e((x^a,y^b)) = |\mathscr{B}_0| = |\mathscr{B}'_0|$ by Theorem 5.11. Elements of $\mathscr{B}_0$ are outside (X,Y) by construction, so $\mathscr{B}_0$ is a k-basis for $R/(X,Y)$. But the same is true for $\mathscr{B}'_0$ and there is only one k-basis consisting of monomials.

$(ii) \Rightarrow (iii)$: Suppose $\mathscr{B}_0 = \mathscr{B}'_0$. For each congruence class in H, $\mathscr{B}_0$ and $\mathscr{B}'_0$ contain exactly one element whose log lies in that class. Since $\mu_{p,q} \in \mathscr{B}'_0$, it must be that $\mu_{p,q} = \mu'_{p,q}$.

$(iii) \Rightarrow (i)$: Suppose $\mu_{p,q} = \mu'_{p,q}$, so that $\mu_{p,q}$ has the smallest x-exponent and the smallest y-exponent of any monomial in its congruence class. In this case, $B_n \cap \{\beta | \log(\beta) = (p,q)\} = \emptyset$. Since this holds for all $(p,q) \in H$, $B_n = \emptyset$ and R is CM by Proposition 5.14. $\square$

5.2 Semigroup rings with four generators

In this section, we will consider semigroup rings of the form $R = k[x^d, x^e y^\ell, x^f y^m, y^n]$ with $d, n > 0$, $e, f, \ell, m \geq 0$ and $(e, \ell) \neq (0,0)$. Our first main result in this section is Theorem 5.28, which gives a simple criterion to determine whether R is Cohen-Macaulay. The second main result is Theorem 5.31, which gives an algorithm to generate a k-basis of $R/(x^d, y^n)$. As noted in Remark 5.5, one may assume that $d = n$ for most results in this section, whereas Corollary 5.34 is probably best stated without assuming $d = n$.

Notation 5.16. Given a group G and an element $g \in G$, we write $\operatorname{ord}(g, G)$ to denote the order of g in G. For elements $(g,h), (g',h') \in \mathbb{Z}^2$ we let $\prec$ denote the reverse lexicographic order.

Throughout this section, we fix $a_i, b_i \in \mathbb{N}$ and $(g_i, h_i) \in d\mathbb{Z} \oplus n\mathbb{Z}$, $i = 1,2,3$ as follows. Let $(g_1, h_1) \in d\mathbb{Z} \oplus n\mathbb{Z}$ be the smallest element with respect to $\prec$ such that there are positive integers a_1, b_1 with $b_2 \geq b_1$ (b_2 to be defined below) and

$$a_1(e,\ell) + b_1(f,m) = (g_1,h_1) \tag{5.1}$$

Let b_2 be the smallest positive integer such that there exist $a_2 \geq 0$ and $(g_2, h_2) \in d\mathbb{Z} \oplus n\mathbb{Z}$ with either at least one of g_2, h_2 being positive or $(g_2, h_2) = (0,0)$ such that

$$-a_2(e,\ell) + b_2(f,m) = (g_2,h_2) \tag{5.2}$$

We choose $a_2 < \operatorname{ord}((e,\ell), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$.

Let a_3 be the smallest positive integer such that there exist $b_3 \geq 0$ and $(g_3, h_3) \in d\mathbb{Z} \oplus n\mathbb{Z}$ with $g_3, h_3 \geq 0$ and g_3, h_3 not both 0 such that

$$a_3(e,\ell) - b_3(f,m) = (g_3,h_3) \tag{5.3}$$

We choose $b_3 < \operatorname{ord}((f,m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$.

Example 5.17. Let $R = k[x^2, x^7y, xy^8, y^3]$. Each (g_i, h_i) must lie in $2\mathbb{Z} \oplus 3\mathbb{Z}$. The three equations defined above are:

$$(7,1) + (1,8) = (8,9)$$
$$-5(7,1) + (1,8) = (-34,3)$$
$$6(7,1) + 0(1,8) = (42,6)$$

Lemma 5.18. *We have $a_3 > a_2$ and $b_2 > b_3$.*

Proof. If $a_3 \leq a_2$, then (5.2) + (5.3) gives

$$(b_2 - b_3)(f,m) = (g_2 + g_3, h_2 + h_3) + (a_2 - a_3)(e,\ell)$$

By the definitions of b_2 and a_3, at least one of $g_2 + g_3$ or $h_2 + h_3$ is positive, so $b_2 - b_3 > 0$. If $b_3 = 0$, then the definition of a_3 gives $\operatorname{ord}((e,\ell),(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) = a_3 \leq a_2$, contradicting the choice of a_2, so $b_3 > 0$. But then $b_2 > b_2 - b_3 > 0$ contradicts the minimality of b_2 in (5.2). Therefore $a_3 > a_2$.

If $b_2 \leq b_3$, then (5.2) + (5.3) gives

$$(a_3 - a_2)(e,\ell) = (g_2 + g_3, h_2 + h_3) + (b_3 - b_2)(f,m)$$

Again at least one of $g_2 + g_3$ or $h_2 + h_3$ is positive, so $a_3 - a_2 > 0$. If $a_2 = 0$, then the definition of b_2 gives $\operatorname{ord}((f,m),(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) = b_2 \leq b_3$, contradicting the choice of b_3, so $a_2 > 0$. If $g_2 + g_3$ and $h_2 + h_3$ are both nonnegative, then $a_3 > a_3 - a_2 > 0$ contradicts the minimality of a_3 in (5.3). So without loss of generality, suppose that $h_2 + h_3 > 0$ and $g_2 + g_3 < 0$, so $g_2 < 0$ and $h_2 > 0$. Let $q \in \mathbb{Z}$, $q > 1$ be such that $a_3 - (q-1)a_2 > 0$ but $a_3 - qa_2 \leq 0$. Then $(q-1)(5.2) + (5.3)$ gives

$$(a_3 - (q-1)a_2)(e,\ell) = ((q-1)g_2 + g_3, (q-1)h_2 + h_3) + (b_3 - (q-1)b_2)(f,m)$$

Then $(q-1)g_2+g_3<0$ gives $b_3-(q-1)b_2>0$. Next, $q(5.2)+(5.3)$ gives

$$(qb_2-b_3)(f,m)=(qg_2+g_3,qh_2+h_3)+(qa_2-a_3)(e,\ell)$$

Since $qh_2+h_3>0$ and $qa_2-a_3\geq 0$ we have $qb_2-b_3>0$.

Then $qb_2-b_3=b_2-(b_3-(q-1)b_2)<b_2$ contradicts the minimality of b_2 in (5.2). Therefore $b_2>b_3$. $\square$

Lemma 5.19. *If $u,v\in\mathbb{Z}$ are such that $a_3>u\geq 0$, $b_2>v\geq 0$ and $(u,v)\neq(0,0)$, then $u(e,\ell)-v(f,m)\notin d\mathbb{Z}\oplus n\mathbb{Z}$.*

Proof. Suppose that $u(e,\ell)=(g,h)+v(f,m)$ for some $(g,h)\in d\mathbb{Z}\oplus n\mathbb{Z}$. If $g,h\geq 0$ and g,h are not both 0, then $u>0$, contradicting the minimality of a_3.

Otherwise we have $v(f,m)=(-g,-h)+u(e,\ell)$ with $-g>0$, $-h>0$ or $(-g,-h)=(0,0)$, contradicting the minimality of b_2. Therefore such (g,h) does not exist. $\square$

Lemma 5.20. *Suppose that $a,b\in\mathbb{N}$ and $(g,h)\in d\mathbb{Z}\oplus n\mathbb{Z}$ are such that*

$$a(e,\ell)+b(f,m)=(g,h) \tag{5.4}$$

(i) *If $a_2=a=0$, then $b_2\mid b$. If $b_3=b=0$, then $a_3\mid a$.*

(ii) *If $a\geq a_3$ and $b>b_2$, then $g\geq g_2+g_3$ and $h\geq h_2+h_3$. If $a>a_3-a_2$ or $(f,m)\neq(0,0)$, then $(g,h)\neq(g_2+g_3,h_2+h_3)$.*

(iii) *Suppose that $a\geq a_3$, $b\leq b_2$ and $(a,b)\neq(a_3-a_2,b_2-b_3)$. If either $b\geq b_2-b_3$ or $b_2-b_3>b$ and b_3,b are not both 0, then $g\geq g_2+g_3$, $h\geq h_2+h_3$ and $(g,h)\neq(g_2+g_3,h_2+h_3)$.*

(iv) *If $a\leq a_3$, $b\leq b_2$, a_2,a are not both 0 and b_3,b are not both 0, then $(a,b)=(a_3-a_2,b_2-b_3)$ or $(0,0)$, or $a>a_3-a_2$ and $b>b_2-b_3$. In fact, there exists $q\in\mathbb{N}$ such that $a=q(a_3-a_2)$ and $b=q(b_2-b_3)$.*

In particular, we have $(5.1)=(5.2)+(5.3)$.

Proof. (i): If $a_2 = 0$, then $b_2 = \mathrm{ord}((f,m),(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$, so if $a = 0$, then $b_2 \mid b$. Similarly, if $b_3 = b = 0$, then $a_3 = \mathrm{ord}((e,\ell),(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) \mid a$.

(ii) and (iii): Now since $a_3 > a_2$ and $b_2 > b_3$, $(5.2)+(5.3)$ gives

$$(a_3 - a_2)(e,\ell) + (b_2 - b_3)(f,m) = (g_2 + g_3, h_2 + h_3) \tag{5.5}$$

Suppose that $a \geq a_3 \geq a_3 - a_2$ and $b \geq b_2 - b_3$. Then $(5.4) - (5.5)$ gives $g \geq g_2 + g_3$ and $h \geq h_2 + h_3$. Suppose furthermore that $(a,b) \neq (a_3 - a_2, b_2 - b_3)$. If $a > a_3 - a_2$, then $g > g_2 + g_3$ or $h > h_2 + h_3$. If $a = a_3 - a_2$ so that $b > b_2 - b_3$, then $b_2 \leq \mathrm{ord}((f,m),(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) \mid b - (b_2 - b_3)$, so $b > b_2$. So suppose that $a \geq a_3$, $b < b_2 - b_3$ and b_3, b are not both 0. If $g < g_2 + g_3$ or $h < h_2 + h_3$ or $(g,h) = (g_2 + g_3, h_2 + h_3)$, then $(5.5) - (5.4)$ gives

$$(b_2 - b_3 - b)(f,m) = (g_2 + g_3 - g, h_2 + h_3 - h) + (a - a_3 + a_2)(e,\ell),$$

contradicting the minimality of b_2. So $g \geq g_2 + g_3$, $h \geq h_2 + h_3$ and $(g,h) \neq (g_2 + g_3, h_2 + h_3)$.

(iv): Now suppose that $a_3 \geq a$, $b_2 \geq b$, that a_2, a are not both 0, that b_3, b are not both 0, that $(a,b) \neq (a_3 - a_2, b_2 - b_3)$ and that $(a,b) \neq (0,0)$. Then by Lemma 5.19, we cannot have $a \geq a_3 - a_2$ and $b \leq b_2 - b_3$, or $a \leq a_3 - a_2$ and $b \geq b_2 - b_3$. So suppose that $a < a_3 - a_2$ and $b < b_2 - b_3$. If $a \neq 0$, then $(5.3) - (5.4)$ gives

$$(a_3 - a)(e,\ell) - (b + b_3)(f,m) = (g_3 - g, h_3 - h),$$

contradicting Lemma 5.19. Similarly, $b \neq 0$ and $(5.4) - (5.2)$ gives a contradiction. Therefore $a > a_3 - a_2$ and $b > b_3 - b_2$.

In such a case, let $q \in \mathbb{N}$ be such that $a - (q-1)(a_3 - a_2), b - (q-1)(b_2 - b_3) > 0$ but one of $a - q(a_3 - a_2)$ or $b - q(b_2 - b_3)$ is nonpositive.

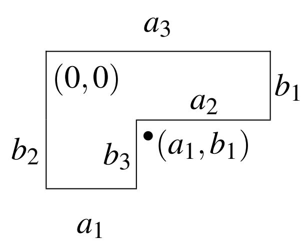
Therefore, $(a - (q-1)(a_3 - a_2), b - (q-1)(b_2 - b_3)) = (a_3 - a_2, b_2 - b_3)$, so $a = q(a_3 - a_2)$ and $b = q(b_2 - b_3)$. $\square$

Notation 5.21. Let a, b denote natural numbers. We let

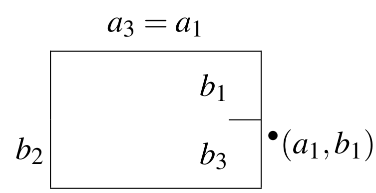
$$\begin{aligned} B_0 &= \{(a,b) \mid a < a_1 \text{ and } b < b_2\} \cup \{(a,b) \mid a < a_3 \text{ and } b < b_1\} \\ &= \{(a,b) \mid a < a_3 \text{ and } b < b_2\} \setminus \{(a,b) \mid a \geq a_1 = a_3 - a_2 \text{ and } b \geq b_1 = b_2 - b_3\} \end{aligned}$$

Let us write $\langle a,b\rangle = a(e,\ell)+b(f,m)$. We write $\langle a,b\rangle \equiv \langle a',b'\rangle$ to mean $\langle a,b\rangle - \langle a',b'\rangle \in d\mathbb{Z}\oplus n\mathbb{Z}$. We let H be the subgroup of $(\mathbb{Z}/d\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})$ generated by $(e,\ell) = \langle 1,0\rangle$ and $(f,m) = \langle 0,1\rangle$.

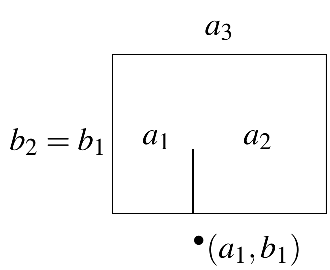
Remark 5.22. We may visualize the set B_0 as follows. For $(a,b) \in \mathbb{N}\times\mathbb{N}$, the first coordinate a increases to the right and the second coordinate b increases downwards.



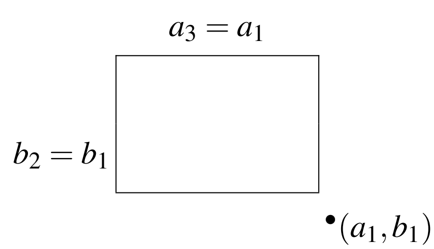
General case



Case $a_2 = 0$, $b_3 \neq 0$



Case $a_2 \neq 0$, $b_3 = 0$



Case $a_2 = b_3 = 0$

Lemma 5.23. *We have* $|B_0| = |H|$.

Proof. $|B_0| \geq |H|$: We will show that for every $\langle a',b'\rangle$ with $a',b' \in \mathbb{N}$, there exists $(a,b) \in B_0$ such that $\langle a,b\rangle \equiv \langle a',b'\rangle$. First, we show that there exist $a'',b'' \in \mathbb{N}$ such that $\langle a',b'\rangle \equiv \langle a'',b''\rangle$ and $b'' < b_2$. Let $q,r \in \mathbb{N}$ be such that $b' = qb_2 + r$ as in the Euclidean algorithm. Then from (5.2) we have $\langle 0,b_2\rangle \equiv \langle a_2,0\rangle$, so $\langle a',b'\rangle \equiv \langle a'+qa_2,r\rangle$ with $b_2 > r \geq 0$.

So assume that $b' < b_2$. We will now reduce to the case that $a' < a_3$. It suffices to show that if $a' \geq a_3$, then there exist $a'',b'' \in \mathbb{N}$ such that $\langle a',b'\rangle \equiv \langle a'',b''\rangle$, $a'' < a'$ and $b'' < b_2$.

Case 1: $b' \geq b_1$

From (5.1) we have $\langle a_1, b_1 \rangle \equiv \langle 0, 0 \rangle$, so $\langle a', b' \rangle \equiv \langle a' - a_1, b' - b_1 \rangle$ with $a' > a' - a_1 \geq a' - a_3 \geq 0$ and $b_2 > b' > b' - b_1 \geq 0$.

Case 2: $b' < b_1$ and $b' + b_3 < b_2$

From (5.3) we have $\langle a_3, 0 \rangle \equiv \langle 0, b_3 \rangle$, so $\langle a', b' \rangle \equiv \langle a' - a_3, b' + b_3 \rangle$.

Case 3: $b' < b_1$ and $b' + b_3 \geq b_2$

From (5.3) and (5.2) we have $\langle a', b' \rangle \equiv \langle a' - a_3 + a_2, b' + b_3 - b_2 \rangle$ with $a' > a' - a_1 = a' - a_3 + a_2$ and $b_2 > b' > b' - b_1 = b' + b_3 - b_2 \geq 0$.

So suppose that $a' < a_3$ and $b' < b_2$ but that $a' \geq a_1$ and $b' \geq b_1$. Now choose $q \in \mathbb{N}$ such that $a' - qa_1, b' - qb_1 \geq 0$ but $a' - (q+1)a_1$ or $b' - (q+1)b_1$ is negative, so that $a' - qa_1 < a_1$ or $b' - qb_1 < b_1$. Then $\langle a', b' \rangle \equiv \langle a' - qa_1, b' - qb_1 \rangle$ and $(a' - qa_1, b' - qb_1) \in B_0$.

$|B_0| \leq |H|$: Suppose that $(a,b), (a',b') \in B_0$ and $(a,b) \neq (a',b')$. If $a' - a \geq 0$ and $b' - b \leq 0$ then $\langle a, b \rangle \not\equiv \langle a', b' \rangle$ by Lemma 5.19. If $a' - a, b' - b \geq 0$ then $\langle a, b \rangle \not\equiv \langle a', b' \rangle$ by Lemma 5.20. Therefore $|B_0| \leq |H|$. $\square$

Notation 5.24. Given $a, b \in \mathbb{N}$, we define the monomial

$$\vec{x}^{\langle a,b \rangle} = (x^e y^\ell)^a (x^f y^m)^b = x^{ae+bf} y^{a\ell+bm}$$

We also define the set of monomials $\mathscr{B}_0 = \{\vec{x}^{\langle a,b \rangle} \mid (a,b) \in B_0\}$.

Remark 5.25. Let $a, b, a', b' \in \mathbb{N}$.

(i) If $a' \geq a$, $b' \geq b$ and $\langle a', b' \rangle - \langle a, b \rangle = (g, h)$, then $g, h \geq 0$.

(ii) Equations (5.1) and (5.3) show that $\vec{x}^{\langle a_1, b_1 \rangle} \in (x^d, y^n)$ and $\vec{x}^{\langle a_3, 0 \rangle} \in \vec{x}^{\langle 0, b_3 \rangle}(x^d, y^n)$. Hence $\vec{x}^{\langle a', b' \rangle} \in (x^d, y^n)$ if $a' \geq a_3$, or $a' \geq a_1$ and $b' \geq b_1$.

(iii) If $a' \leq a$, $b' \leq b$ and $(a,b) \in B_0$, then $(a', b') \in B_0$.

Lemma 5.26. *Given a set $S \subseteq \mathbb{N} \times \mathbb{N}$, the set of monomials $\{\vec{x}^{\langle a,b \rangle} \mid (a,b) \in S\}$ is linearly independent in $R/(x^d, y^n)$ over k if and only if:*

(i) *if $(a,b)\in S$, $a',b'\in\mathbb{N}$ and $\langle a,b\rangle-\langle a',b'\rangle=(g,h)\in d\mathbb{Z}\oplus n\mathbb{Z}$, then $g<0$ or $h<0$ or $(g,h)=(0,0)$, and*

(ii) *if $(a,b),(a',b')\in S$ and $(a,b)\neq(a',b')$, then $\langle a,b\rangle\neq\langle a',b'\rangle$.*

Proof. Every monomial in (x^d,y^n) can be written as a scalar multiple of $x^g\vec{x}^{\langle a,b\rangle}y^h$ for some $a,b\in\mathbb{N}$ and $(g,h)\in d\mathbb{Z}\oplus n\mathbb{Z}$ with $g,h\geq 0$ and $(g,h)\neq(0,0)$. □

Lemma 5.27. *The set $\mathscr{B}_0$ is linearly independent in $R/(x^d,y^n)$ over k.*

Proof. By Lemma 5.23, we only need to verify (i) in Lemma 5.26 for $(a,b)\in B_0$ and $(a',b')\notin B_0$. By Remark 5.25, we may assume that $a'<a$ or $b'<b$. If $a'<a$, then by Remark 5.25 we have $b'>b$, so $\langle a-a',0\rangle=\langle 0,b'-b\rangle+(g,h)$. By the minimality of a_3 we have $g<0$, $h<0$ or $(g,h)=(0,0)$. Similarly, if $b'<b$, then $a'>a$ and $\langle 0,b-b'\rangle=\langle a'-a,0\rangle+(g,h)$ and the result follows from the minimality of b_2. □

Theorem 5.28. *For the ring $R=k[x^d,x^ey^\ell,x^fy^m,y^n]$, we have:*

(i) $|\mathscr{B}_0|=|H|=\begin{vmatrix}a_3 & -b_3\\ -a_2 & b_2\end{vmatrix}=\begin{vmatrix}a_3 & -b_3\\ a_1 & b_1\end{vmatrix}=\begin{vmatrix}a_1 & b_1\\ -a_2 & b_2\end{vmatrix}$

(ii) $\dim_k R/(x^d,y^n)\geq|H|=|\mathscr{B}_0|$

(iii) *The ring R is Cohen-Macaulay iff $\mathscr{B}_0$ is a basis of $R/(x^d,y^n)$ over k iff $g_2,h_2\geq 0$.*

Proof. (i): We have $|\mathscr{B}_0|=|B_0|=|H|$ by Lemma 5.23. The definition of B_0 gives

$$|B_0|=a_3b_2-(a_3-a_1)(b_2-b_1)=a_3b_2-a_2b_3$$

The rest again follows from $(5.1)=(5.2)+(5.3)$.

(ii): By Lemma 5.27, the set $\mathscr{B}_0$ is linearly independent in $R/(x^d,y^n)$ over k.

(iii): If $g_2<0$ or $h_2<0$, then $(a,b)=(0,b_2)$ and $(a',b')=(a_2,0)$ satisfy Lemma 5.26 by (5.2). Let us verify Lemma 5.26(i) for $(0,b_2)$ and $(a',b')\notin B_0$. By Remark 5.25 we may assume that $b'<b_2$. If $b'>0$, then $\langle 0,b_2-b'\rangle=\langle a',0\rangle+(g,h)$ and (i) is satisfied by the linear independence of $\mathscr{B}_0$ in $R/(x^d,y^n)$ over k. If $b'=0$, then $a'\geq a_3>a_2$. If $\langle 0,b_2\rangle=\langle a',0\rangle+(g,h)$ with $g,h\geq 0$, then $g_2, h_2 \geq 0$ in (5.2), contradicting our assumption. Therefore Lemma 5.26(i) holds for $(a,b) = (0,b_2)$ and hence $\mathscr{B}_0 \cup \{\vec{x}^{\langle 0,b_2\rangle}\}$ is linearly independent in $R/(x^d, y^n)$ over k.

If $g_2, h_2 \geq 0$, then $\vec{x}^{\langle 0,b_2\rangle} = \vec{x}^{\langle a_2,0\rangle}$ or $\vec{x}^{\langle 0,b_2\rangle} \in \vec{x}^{\langle a_2,0\rangle}(x^d, y^n)$. In the first case, for $a, b \in \mathbb{N}$ we have $\vec{x}^{\langle a,b\rangle} = \vec{x}^{\langle a+qa_2, b-qb_2\rangle}$ for any $q \in \mathbb{Z}$. So by the definition of B_0 and Remark 5.25 we see that for all $(a',b') \notin B_0$ either $\vec{x}^{\langle a',b'\rangle} \in (x^d, y^n)$ or $\vec{x}^{\langle a',b'\rangle} = \vec{x}^{\langle a,b\rangle}$ for some $(a,b) \in B_0$. Therefore $\mathscr{B}_0$ is a basis of $R/(x^d, y^n)$ over k.

Finally, Theorems 5.4 and 5.11 show that R is Cohen-Macaulay iff $\dim_k R/(x^d, y^n) = |H|$. By (ii), $\dim_k R/(x^d, y^n) = |H|$ iff $\mathscr{B}_0$ is a basis of $R/(x^d, y^n)$ over k iff $g_2, h_2 \geq 0$. $\square$

Remark 5.29. In part (iii) of Theorem 5.28, instead of using Theorems 5.4 and 5.11, one can also prove the result using the fact that R is Cohen-Macaulay iff x^d, y^n is a regular sequence.

Corollary 5.30. *The ring $k[x^d, x^e y^\ell, y^n]$ is Cohen-Macaulay, where $d, n > 0$ and $(e,\ell) \neq (0,0)$.*

Proof. Take $(f,m) = u_1(d,0) + u_2(e,\ell) + u_3(0,n)$ for any $u_1, u_2, u_3 \in \mathbb{N}$. $\square$

Theorem 5.31. *We can use the following algorithm to obtain a basis of $R/(x^d, y^n)$ over k.*

1 Let $B = B_0$.

2 Let base $= a_1$, $a^* = a_2$, $b^* = b_2$, $g^* = g_2$ and $h^* = h_2$.

3 While $g^* < 0$ or $h^* < 0$, do the following steps.

4 If $a^* \geq a_1$, then:

Replace B by $B \cup \{(0,b^*) + (u,v) \mid u <$ base and $v < b_1\}$.

Replace a^* by $a^* - a_1$, b^* by $b^* + b_1$, g^* by $g^* + g_1$ and h^* by $h^* + h_1$.

5 If $a^* \leq a_1 -$ base, then:

Replace B by $B \cup \{(0,b^*) + (u,v) \mid u <$ base and $v < b_2\}$.

Replace a^* by $a^* + a_2$, b^* by $b^* + b_2$, g^* by $g^* + g_2$ and h^* by $h^* + h_2$.

6 If $a_1 -$ base $< a^* < a_1$, then:

Replace B by $B \cup \{(0,b^*) + (u,v) \mid (u <$ base and $v < b_1)$ or $(u < a_1 - a^*$ and $v < b_2)\}$.

Replace a^* by $a^* + a_2$, b by $b^* + b_2$, g^* by $g^* + g_2$, h^* by $h^* + h_2$ and base by $a_1 - a^*$.

After the algorithm stops, the set of monomials $\mathscr{B} = \{\vec{x}^{\langle a,b\rangle} \mid (a,b) \in B\}$ *forms a basis of* $R/(x^d, y^n)$ *over* k.

Remark 5.32. Theorem 5.31 only needs to use information from (5.1) and (5.2), or equivalently, from (5.2) and (5.3). Given the equation

$$-a^*(e,\ell) + b^*(f,m) = (g^*,h^*), \tag{5.6}$$

Step 4 corresponds to $(5.6)+(5.1)$ and Steps 5 and 6 correspond to $(5.6)+(5.2)$. Furthermore, in each iteration of the algorithm, the new elements added to the set B are in one-to-one correspondence with those in $\{(a,b) \in B_0 \mid a^* \le a < a^* + base\}$.

Proof of Theorem 5.31. First, we note by induction that throughout the algorithm,

(a) $a^* + base \le a_3$,

(b) the value of *base* is always positive and weakly decreasing, and

(c) if $a,b,a',b' \in \mathbb{N}$, $a' \le a$, $b' \le b$ and $(a,b) \in B$, then $(a',b') \in B$.

Let ^{u} denote the updated value of a variable after an iteration of the algorithm. We note also that in Steps 4, 5 and 6:

(d) Let $C = B^u \setminus B$ and $(a,b) \in C$. Then $\langle a,b\rangle \equiv \langle a+a^*, b-b^*\rangle$ and $(a+a^*, b-b^*) \in B_0$. Hence if $(a',b') \in C$ such that $(a,b) \ne (a',b')$, then $\langle a,b\rangle \not\equiv \langle a',b'\rangle$.

We will now prove by induction on the number of iterations that after each iteration of the algorithm,

(e) the set $\mathscr{B}^u = \{\vec{x}^{\langle a,b\rangle} \mid (a,b) \in B^u\}$ is linearly independent in $R/(x^d, y^n)$ over k, and

(f) $\vec{x}^{\langle a',b'\rangle} \in (x^d, y^n)$ for all $(a',b') \notin B^u$ such that $b' < b^{*u}$.

The base case of $B = \emptyset$, i.e. $B^u = B_0$, is given by Theorem 5.28(ii) and Remark 5.25. In the induction step, we will first show (e) by using Lemma 5.26.

Let $(a,b) \in C = B^u \setminus B$ and $(a',b') \in B$ such that $\langle a,b\rangle \equiv \langle a',b'\rangle$. If $(a',b') \in B_0$, then we find $(a',b') = (a+a^*, b-b^*)$ and $\langle a,b\rangle - \langle a+a^*, b-b^*\rangle = (g^*,h^*)$. By assumption, $g^* < 0$ or $h^* < 0$, so (a,b) and (a',b') satisfy Lemma 5.26. If $(a',b') \notin B_0$, then we have $\langle a, b-b_2\rangle \equiv \langle a', b'-b_2\rangle$ and by the linear independence of $\mathscr{B}$, (a,b) and (a',b') again satisfy Lemma 5.26.

So suppose that $(a',b') \notin B$ and $\langle a,b\rangle - \langle a',b'\rangle = (g,h) \in d\mathbb{Z} \oplus n\mathbb{Z}$. We verify Lemma 5.26(i). By the proof of Lemma 5.27, we may assume that $b' < b$.

Case 1: $b' \geq b_2$

Lemma 5.26(i) holds from $\langle a, b-b'\rangle - \langle a', 0\rangle = (g,h)$ and the linear independence of $\mathscr{B}$.

Case 2: $b_2 > b' \geq b_1$

By the definition of B_0 we have $a' \geq a_1$. Let $q \in \mathbb{N}$ be such that $a'-qa_1, b'-qb_1 \geq 0$ but one of $a'-(q+1)a_1$ or $b'-(q+1)b_1$ is negative. Thus

$$\langle a,b\rangle - \langle a'-qa_1, b'-qb_1\rangle = \langle a,b\rangle - \langle a',b'\rangle + q\langle a_1,b_1\rangle = (g+qg_1, h+qh_1)$$

If $(a'-qa_1, b'-qb_1) \in B_0$, then $g+qg_1 = g^* < 0$ or $h+qh_1 = h^* < 0$, so Lemma 5.26(i) holds for (a,b) and (a',b'). Otherwise, replacing $(a'-qa_1, b'-qb_1)$ by (a',b'), we are reduced to the case where $b' < b_1$.

Case 3: $b_1 > b'$

By the definition of B_0 we have $a' \geq a_3$. In this case,

$$\langle a',b'\rangle - \langle a+a^*, b-b^*\rangle = \langle a',b'\rangle - \langle a,b\rangle + \langle a,b\rangle - \langle a+a^*, b-b^*\rangle = (g^*-g, h^*-h)$$

If $b' \geq b-b^*$, then $0 > g^* \geq g$ or $0 > h^* \geq h$ and Lemma 5.26(i) is satisfied. If $b' < b-b^*$, then $\langle 0, b-b^*-b'\rangle = \langle a'-(a+a^*), 0\rangle + (g-g^*, h-h^*)$. By the minimality of b_2 we have $g-g^*$ and $h-h^* \leq 0$ and again we are done.

Now we verify (f). Let $(a',b') \notin B^u$ with $b' < b^{*u}$. By induction, we may assume that $b' \geq b^*$ and by Remark 5.25 we may assume that $a' < base$, so we only need to consider Step 6 with $a' \geq a_1 - a^*$ and $b' \geq b^* + b_1$. We have $\langle a_1 - a^*, b^* + b_1\rangle = \langle a_1, b_1\rangle + (g^*, h^*) \in d\mathbb{Z} \oplus n\mathbb{Z}$, so $\vec{x}^{\langle a_1 - a^*, b^*+b_1\rangle} \in (x^d, y^n)$ and the result follows from Remark 5.25.

Finally, the algorithm must stop at or before $b^* = \operatorname{ord}((f,m),(\mathbb{Z}/d\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z}))$. After the algorithm stops, we already know that $\mathscr{B}$ is linearly independent by (e). Moreover by (5.6) we have $\vec{x}^{\langle 0,b^*\rangle} = \vec{x}^{\langle a^*,0\rangle}$ or $\vec{x}^{\langle 0,b^*\rangle} \in \vec{x}^{\langle a^*,0\rangle}(x^d,y^n)$. By (f) and Remark 5.25 we see that for all $(a',b') \notin B$ either $\vec{x}^{\langle a',b'\rangle} \in (x^d,y^n)$ or $\vec{x}^{\langle a',b'\rangle} = \vec{x}^{\langle a,b\rangle}$ for some $(a,b)\in B$. Therefore $\mathscr{B}$ is a basis of $R/(x^d,y^n)$ over k. □

Remark 5.33. Each iteration of the algorithm gives $\langle 0,b^*\rangle \equiv \langle a^*,0\rangle$. Since the algorithm must stop at or before $b^* = \operatorname{ord}((f,m),(\mathbb{Z}/d\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z}))$, we cannot have $\langle 0,b^{1*}\rangle \equiv \langle 0,b^{2*}\rangle$ for different values b^{1*},b^{2*} of b^*. Hence $a_3 \leq \operatorname{ord}((e,\ell),(\mathbb{Z}/d\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})) \leq |H| \leq dn$ is an upper bound on the number of iterations of the algorithm.

Corollary 5.34. *We have* $\dim_k R/(x^d,y^n) \leq |H|(|H|+1)/2 \leq dn(dn+1)/2$.

Proof. Let us set $a^* = 0$ in Step 1. In each iteration, at most $i_{a^*} = |\{(a,b)\in B_0 \mid a \geq a^*\}|$ elements are added to the set B by Remark 5.32. We have $1 \leq i_{a^*} \leq |B_0| = |H| \leq dn$ and that the map $a^* \mapsto i_{a^*}$ is injective. Since there exist at most $|H|$ possible values of a^* before the stopping criterion is reached, we have $\dim_k R/(x^d,y^n) = |B| \leq \sum_{i=1}^{|H|} i = |H|(|H|+1)/2 \leq dn(dn+1)/2$. □

Example 5.35. Here we give examples showing that the algorithm is "best possible", in the sense that the maximum number of iterations can be attained. In the second example, we will show that the upper bound in Corollary 5.34 is also attained. Let p,q be distinct prime numbers.

(a) Let $R = k[x^p, x^{jpq-1}y, xy^{jpq-1}, y^q]$ with $j\in\mathbb{N}$, $j>1$. The successive values of $\langle 0,b^*\rangle$ are

$$(1, jpq-1),\ (2,2(jpq-1)),\ \ldots,\ (pq-1,(pq-1)(jpq-1)),\ (pq,pq(jpq-1))$$

and those of $\langle a^*,0\rangle$ are

$$((pq-1)(jpq-1),pq-1),\ ((pq-2)(jpq-1),pq-2),\ \ldots,\ (jpq-1,1),\ (0,0)$$

When $j=2$, $p=2$ and $q=3$, we display the elements of $\langle B\rangle = \{\langle a,b\rangle \mid (a,b)\in B\} = \log(\mathscr{B})$ as follows.

(0,0)	(11,1)	(22,2)	(33,3)	(44,4)	(55,5)
(1,11)					
(2,22)					
(3,33)					
(4,44)					
(5,55)					

$$R = k[x^2, x^{11}y, xy^{11}, y^3]$$

(b) Let $R = k[x^p, x^{jpq+1}y, xy^{jpq+1}, y^q]$ with $j \in \mathbb{N}$, $j > 0$. The successive values of $\langle 0, b^* \rangle$ are

$$(1, jpq+1),\ (2, 2(jpq+1)),\ \ldots,\ (pq-1, (pq-1)(jpq+1)),\ (pq, pq(jpq+1))$$

and those of $\langle a^*, 0 \rangle$ are

$$(jpq+1, 1),\ (2(jpq+1), 2),\ \ldots,\ ((pq-1)(jpq+1), pq-1),\ (0,0)$$

When $j = 1$, $p = 2$ and $q = 3$, we display the elements of $\langle B \rangle$ as follows.

(0,0)	(7,1)	(14,2)	(21,3)	(28,4)	(35,5)
(1,7)	(8,8)	(15,9)	(22,10)	(29,11)	
(2,14)	(9,15)	(16,16)	(23,17)		
(3,21)	(10,22)	(17,23)			
(4,28)	(11,29)				
(5,35)					

$$R = k[x^2, x^7y, xy^7, y^3]$$

Remark 5.36. Having found the basis $\mathscr{B}$ of $R/(x^a, y^b)$ as in Theorem 5.31, one may sort the monomials in $\mathscr{B}$ and find the Hilbert polynomial $P(n)$ for (x^a, y^b) by Theorem 5.11 and the least integer m such that the Hilbert polynomial equals the Hilbert function for all $n \geq m$ by Proposition 5.10.

5.3 Projective monomial curves in $\mathbb{P}^3$

In this section, we consider rings of the form $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ with $0 < \ell < m < n$. We will apply the results from Section 5.2 to obtain stronger results for such rings R. In particular, Theorem 5.42 gives a simple criterion to determine whether R is Cohen-Macaulay and Theorem 5.46 gives a simple algorithm to generate a k-basis of $R/(x^n, y^n)$.

Notation 5.37. In this section, we fix $a_i, b_i, c_i, h_i \in \mathbb{N}$, $i = 1,2,3$ as follows. Let c_1 be the smallest integer such that there are $m/\gcd(\ell,m) \geq a_1 > 0$ and $b_1 > 0$ with

$$a_1\ell + b_1 m = c_1 n = h_1 \tag{5.7}$$

Let b_2 be the smallest integer such that there are $n/\gcd(\ell,n) > a_2 \geq 0$ and $c_2 > 0$ with

$$-a_2\ell + b_2 m = c_2 n = h_2 \tag{5.8}$$

Let a_3 be the smallest positive integer such that there are $n/\gcd(m,n) > b_3 \geq 0$ and $c_3 \geq 0$ with

$$a_3\ell - b_3 m = c_3 n = h_3 \tag{5.9}$$

Remark 5.38. For any $d \in \mathbb{Z}$ we have

$$\operatorname{ord}((n-d,d),(\mathbb{Z}/n\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})) = \operatorname{ord}((-d,d),(\mathbb{Z}/n\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})) = n/\gcd(d,n)$$

Lemma 5.39. *Let $a,b,c,d \in \mathbb{N}$.*

(i) *If $-a(n-\ell,\ell) + b(n-m,m) = (cn,dn)$, $b > 0$ and $c \geq 0$, then $d > 0$.*

(ii) *If $a(n-\ell,\ell) - b(n-m,m) = (cn,dn)$, $a > 0$ and $d \geq 0$, then $c > 0$.*

Proof. (i): Since $b > 0$ we have $(a,c) \neq (0,0)$. If $c \geq 0$, then $b(n-m) = cn + a(n-\ell)$, but $cn + a(n-\ell) > (c+a)(n-m)$, so $b > c+a$. Hence $dn = -a\ell + bm = (b-a-c)n > 0$.
(ii): Replace a by b, b by a, ℓ by $n-m$, m by $n-\ell$, c by d and d by c in (i). □

Lemma 5.40. *The definitions of a_i, b_i, h_i, $i = 1,2,3$ in Section 5.2 and in this section coincide.*

Proof. Let us temporarily write a_i', b_i', h_i', $i = 1,2,3$ for the definitions of a_i, b_i, h_i in this section. We have $e = n - \ell$ and $f = n - m$. Let us first consider (5.2). Suppose that $g_2 \geq 0$. By Lemma 5.39(i) we have $h_2 > 0$, so the conditions that $g_2 > 0$ or $(g_2, h_2) = (0,0)$ become redundant. Hence $b_2 = b_2'$, $a_2 = a_2'$ and $h_2 = h_2'$.

Similarly, in (5.3) suppose that $h_3 \geq 0$. By Lemma 5.39(ii), $g_3 \geq 0$ and $(g_3, h_3) \neq (0,0)$ are redundant conditions. Hence $a_3 = a_3'$, $b_3 = b_3'$ and $h_3 = h_3'$.

Now $(5.8) + (5.9)$ gives

$$(a_3 - a_2)\ell + (b_2 - b_3)m = (c_2 + c_3)n = h_2 + h_3$$

Let us show that $c_1 = c_2 + c_3$. First we have $a_3 - a_2 \leq a_3 \leq m/\gcd(\ell, m)$. Now suppose that $a, b, c \in \mathbb{N}$ are such that $a, b > 0$, $a\ell + bm = cn$ and $(a,b) \neq (a_3 - a_2, b_2 - b_3)$. By Remark 5.25 we may assume that $a < a_3 - a_2$ or $b < b_2 - b_3$. If $b < b_2 - b_3$, then $c \geq c_2 + c_3$ by Lemma 5.20. If $c = c_2 + c_3$, then $(a - (a_3 - a_2))\ell = (b_2 - b_3 - b)m$, so we have $(m/\gcd(\ell, m)) \mid a - (a_3 - a_2)$ and $a > m/\gcd(\ell, m)$. If $a < a_3 - a_2$, then $b > b_2 - b_3$ by Lemma 5.20 and hence $c > c_2 + c_3$ by the minimality of a_3. Therefore $h_1 = h_1'$, $a_1 = a_1'$ and $b_1 = b_1'$. $\square$

Notation 5.41. We let $B_0, H, \vec{x}^{\langle a,b\rangle}, \mathscr{B}_0$ be as in Section 5.2.

Theorem 5.42. *Let $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ and $d = \gcd(\ell, m, n)$. Then:*

(i) $\quad n = d\begin{vmatrix} a_3 & -b_3 \\ -a_2 & b_2 \end{vmatrix} = d\begin{vmatrix} a_3 & -b_3 \\ a_1 & b_1 \end{vmatrix} = d\begin{vmatrix} a_1 & b_1 \\ -a_2 & b_2 \end{vmatrix}$

(ii) $\quad m = d\begin{vmatrix} a_3 & c_3 \\ -a_2 & c_2 \end{vmatrix} = d\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} = d\begin{vmatrix} a_1 & c_1 \\ -a_2 & c_2 \end{vmatrix}$

(iii) $\quad \ell = d\begin{vmatrix} c_3 & -b_3 \\ c_2 & b_2 \end{vmatrix} = d\begin{vmatrix} c_3 & -b_3 \\ c_1 & b_1 \end{vmatrix} = d\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

(iv) $\dim_k R/(x^d, y^n) \geq n/d = |\mathscr{B}_0|$

(v) *The ring R is Cohen-Macaulay iff $\mathscr{B}_0$ is a basis of $R/(x^d, y^n)$ over k iff $b_2 \geq a_2 + c_2$.*

Proof. We may identify H with the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by ℓ and m, so $|H| = n/d$. Therefore (iv) and (i) follow from (i) of Theorem 5.28, and (ii) and (iii) follow from Cramer's rule. Using (iii) of Theorem 5.28, we have $g_2 \geq 0$ iff $-a_2(n-\ell) + b_2(n-m) \geq 0$ iff $(b_2 - a_2 - c_2)n \geq 0$ iff $b_2 \geq a_2 + c_2$, so (v) follows from Lemma 5.39(i). □

Corollary 5.43. *Let $\ell = 1$ and $n = qm + r$ as in the Euclidean algorithm. If $r = 0$, then R is Cohen-Macaulay. If $r \neq 0$, then R is Cohen-Macaulay if and only if $q + r \geq m$.*

Proof. If $r = 0$, then $a_2 = 0$, $b_2 = q$ and $c_2 = 1$, so $b_2 = q \geq 1 = a_2 + c_2$. If $r \neq 0$, then $a_2 = m - r$, $b_2 = q + 1$ and $c_2 = 1$, so R is Cohen-Macaulay iff $q + 1 \geq m - r + 1$ iff $q + r \geq m$. □

Corollary 5.44. *If $\gcd(\ell, m) = 1$ and $\ell + m = n$, then R is Cohen-Macaulay if and only if $m = \ell + 1$.*

Proof. Since $\gcd(\ell, m) = 1$ and $\ell + m = n$ we have $\gcd(m, n) = 1$. From $b_2 m - a_2(n-m) = c_2 n$ we get $(b_2 + a_2)m = (c_2 + a_2)n$. By the minimality of b_2 we get $b_2 + a_2 = n$ and $c_2 + a_2 = m$, so $c_2 = 1$, $a_2 = m - 1$ and $b_2 = n - (m-1)$. Then $b_2 \geq a_2 + c_2$ iff $n - (m-1) \geq m - 1 + 1$ iff $n \geq 2m - 1$. But $n = m + \ell \leq m + m - 1 = 2m - 1$, so R is Cohen-Macaulay iff $n = 2m - 1$ iff $m = \ell + 1$. □

Remark 5.45. We therefore recover Macaulay's result that $k[x^4, x^3y, xy^3, x^4]$ is not Cohen-Macaulay.

Theorem 5.46. *We can use the following algorithm to obtain a basis of $R/(x^n, y^n)$ over k.*

1 Let $B = B_0$.

2 Let $base = a_1$, $a^* = a_2$, $b^* = b_2$ and $c^* = c_2$.

3 While $b^* < a^* + c^*$, do the following steps.

4 If $a^* \geq a_1$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base \text{ and } v < b_1\}$.

Replace a^* by $a^* - a_1$, b^* by $b^* + b_1$ and c^* by $c^* + c_1$.

5 If $a^* \leq a_1 - base$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base \text{ and } v < b_2\}$.

Replace a^* by $a^* + a_2$, b^* by $b^* + b_2$ and c^* by $c^* + c_2$.

6 If $a_1 - base < a^* < a_1$, then:

Replace B by $B \cup \{(0,b^*) + (u,v) \mid (u < base \text{ and } v < b_1) \text{ or } (u < a_1 - a^* \text{ and } v < b_2)\}$.

Replace a^* by $a^* + a_2$, b by $b^* + b_2$, c^* by $c^* + c_2$ and base by $a_1 - a^*$.

After the algorithm stops, the set of monomials $\mathscr{B} = \{\vec{x}^{\langle a,b \rangle} \mid (a,b) \in B\}$ forms a basis of $R/(x^n, y^n)$ over k. □

Example 5.47. Let $R = k[x^{23}, x^{21}y^2, x^5y^{18}, y^{23}]$. We will use Theorem 5.46 to calculate the size of the monomial k-basis $\mathscr{B}$ of $R/(x^{23}, y^{23})$ and find the elements of $\mathscr{B}$.

Step	$\lvert B \rvert$	*base*	Equation	Remark
			$2 \times 18 = -5 \times 2 + 2 \times 23$ (5.7)	
1	23	5	$3 \times 18 = 4 \times 2 + 2 \times 23$ (5.8)	
6	34	1	$6 \times 18 = 8 \times 2 + 4 \times 23$	Add equation (5.8) (to previous).
4	36	1	$8 \times 18 = 3 \times 2 + 6 \times 23$	Add equation (5.7).
5	39	1	$11 \times 18 = 7 \times 2 + 8 \times 23$	Add equation (5.8).
4	$\boxed{41}$	1	$13 \times 18 = 2 \times 2 + 10 \times 23$	Add equation (5.7) and stop.

We display the second coordinates of $\langle B \rangle = \log(\mathscr{B})$, i.e. the y-degrees of elements of $\mathscr{B}$, as follows.

0	2	4	6	8	10	12	14	16
18	20	22	24	26	28	30	32	34
36	38	40	42	44				
54	56	58	60	62				
72	74	76	78	80				
90								
108								
126								
144								
162								
180								
198								
216								

For a general ring of the form $R[x^d, x^e y^\ell, x^f y^m, y^n]$, it was shown that the bound in Corollary 5.34 is sharp. However, it may easily be deduced, say by observing that $\{(x^{n-\ell} y^\ell)^j | 0 \leq j \leq n-1\}$ is never disjoint from (x^n, y^n), that no projective monomial curve achieves this bound. One may hope, then, to identify a tighter bound in this special case.

Bibliography

[1] Rudiger Achilles and Mirella Manaresi. Multiplicity for ideals of maximal analytic spread and intersection theory. *J. Math. Kyoto Univ.*, 33:1029–1046, 1993.

[2] Wayne Bishop. *A Theory of Multiplicity for Multiplicative Filtrations*. Thesis, Western Michigan University, 1971.

[3] Wayne Bishop, J. W. Petro, L.J. Ratliff, and D.E. Rush. Note on Noetherian filtrations. 17:471–485, 1989.

[4] Winfried Bruns and Jurgen Herzog. *Cohen-Macaulay Rings*. Cambridge University Press, 1998.

[5] David Buchsbaum and Dock S. Rim. A generalized Koszul complex. ii. depth and multiplicity. *Amer. Math. Soc.*, 111:197–224, 1964.

[6] Steven Dale Cutkosky. Asymptotic growth of saturated powers and epsilon multiplicity. *Math. Res. Lett.*, 18:93–106, 2011.

[7] Steven Dale Cutkosky, Huy Tai Ha, Hema Srinivasan, and Emanoil Theodorescu. Asymptotic behavior of the length of local cohomology. *Canad. J. Math.*, 57:1178–1192, 2005.

[8] H. Dichi and D. Sangare. Hilbert functions, Hilbert-Samuel quasi-polynomials with respect to f-good filtrations, multiplicities. *J. Pure Appl. Algebra*, 138:205–213, 1999.

[9] Paul Eakin and Stephen McAdam. The asymptotic *ass*. *J. Algebra*, 61:71–81, 1979.

[10] David Eisenbud, Craig Huneke, and Bernd Ulrich. What is the Rees algebra of a module? *Proc. Amer. Math. Soc.*, 131:701–708, 2003.

[11] Hubert Flenner, Liam O'Carroll, and Wolfgang Vogel. *Joins and Intersections*. Springer, 1999.

[12] S. Goto, N. Suzuki, and K. Watanabe. On affine semigroups. *Japan. J. Math.*, 2:1–12, 1976.

[13] Manfred Herrmann, Eero Hyry, Jurgen Ribbe, and Zhongming Tang. Reduction numbers and multiplicities of multigraded structures. *J. Algebra*, 197:311–341, 1997.

[14] Jurgen Herzog, Tony Puthenpurakal, and Jugal Verma. Hilbert polynomials and powers of ideals. *Math. Proc. Cambridge Philos. Soc.*, 145:623–642, 2008.

[15] Jurgen Herzog, Asia Rauf, and Marius Vladoiu. The stable set of associated prime ideals of a polymatroidal ideal. *J. Algebraic Combin.*, 37:289–312, 2013.

[16] Melvin Hochster. Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *Ann. of Math.*, 96:318–337, 1972.

[17] Craig Huneke and Irena Swanson. *Integral Closure of Ideals, Rings, and Modules*. Cambridge Mathematical Press, 2006.

[18] Jack Jeffries and Jonathan Montano. The j-multiplicity of monomial ideals. *Math. Res. Lett.*, 20:729–744, 2013.

[19] Daniel Katz. Prime divisors, asymptotic r-sequences and unmixed local rings. *J. Algebra*, 95:59–71, 1985.

[20] Daniel Katz and Javid Validashti. Multiplicities and Rees valuations. *Collect. Math.*, 61:1–24, 2010.

[21] David Kirby. On the Buchsbaum-Rim multiplicity associated with a matrix. *J. London Math. Soc.*, 32:57–61, 1985.

[22] David Kirby and David Rees. Multiplicities in graded rings ii: integral equivalence and the Buchsbaum-Rim multiplicity. *Math. Proc. Camb. Phil. Soc*, 119:425–445, 1996.

[23] Steven Kleiman and Anders Thorup. Mixed Buchsbaum-Rim multiplicities. *Amer. J. Math.*, 118:529–569, 1996.

[24] F.S. Macaulay. Algebraic theory of modular systems. *Cambridge Tracts in Math.*, 19, 1916.

[25] Mousaumi Mandal, Balwant Singh, and Jugal Verma. On some conjectures about Chern numbers of filtrations. *J. Algebra*, 325:147–162, 2011.

[26] José Martínez-Bernal, Susan Morey, and Rafael H. Villarreal. Associated primes of powers of edge ideals. *Collect. Math.*, 63:361–374, 2012.

[27] Hideyuki Matsumura. *Commutative Ring Theory*. Cambridge University Press, 1989.

[28] L.J. Ratliff, Jr. and J.S. Okon. Reductions of filtrations. *Pacific Journal of Mathematics*, 144, 1990.

[29] David Rees. Valuations associated with ideals. ii. *J. London Math. Soc.*, 31:221–228, 1956.

[30] David Rees. $\mathfrak{A}$-transforms of local rings and a theorem on multiplicities of ideals. *Proc. Cambridge Phil. Soc.*, 57:8–17, 1961.

[31] David Rees. *Lectures on the Asymptotic Theory of Ideals*. Cambridge University Press, 1988.

[32] L. Reid and L.G. Roberts. Maximal and Cohen-Macaulay projective monomial curves. *J. Algebra*, 307:409–423, 2007.

[33] Glenn Rice. *Asymptotic Properties of Torsion-Free Symmetric Powers of Modules*. Thesis, University of Kansas, 2005.

[34] Paul Roberts. *Multiplicities and Chern Classes in Local Algebra*. Cambridge University Press, 1998.

[35] Pierre Samuel. La notion de multiplicité en algèbre et en géométrie algébrique. ii. *J. Math. Pures Appl.*, 30:207–274, 1951.

[36] Aron Simis, Bernd Ulrich, and Wolmer Vasconcelos. Codimension, multiplicity and integral extensions. *Math. Proc. Cambridge Philos. Soc.*, 130:237–257, 2001.

[37] Aron Simis, Bernd Ulrich, and Wolmer Vasconcelos. Rees algebras of modules. *Proc. London Math. Soc.*, 87:610–646, 2003.

[38] Richard Stanley. Hilbert functions of graded algebras. *Adv. in Math.*, 68:175–193, 1982.

[39] Bernard Teissier. Monomes, volumes et multiplicites. *In 'Introduction a la theorie des singularites, II', Travaux en Cours*, 37:127–141, 1988.

[40] N.V. Trung and L.T. Hoa. Affine semigroups and Cohen-Macaulay rings generated by monomials. *Trans. Amer. Math. Soc.*, 298:145–167, 1986.

[41] Bernd Ulrich and Javid Validashti. A criterion for integral dependence of modules. *Math. Res. Lett.*, 15:149–162, 2008.

[42] Bernd Ulrich and Javid Validashti. Numerical criteria for integral dependence. *Math. Proc. Cambridge Philos. Soc.*, 151:95–102, 2011.

[43] J. K. Verma. Rees algebras and mixed multiplicities. *Proc. Amer. Math. Soc.*, 104:1036–1042, 1988.

[44] Santiago Zarzuela. On the depth of blowup algebras of ideals with analytic deviation one. *Proc. Amer. Math. Soc.*, 123:3639–3647, 1995.